Rethinking Quaternions

Theory and Computation

Synthesis Lectures on Computer Graphics and Animation

Editor
Brian A. Barsky, *University of California, Berkeley*

Rethinking Quaternions
Theory and Computation
Ron Goldman
2010

Information Theory Tools for Computer Graphics
Mateu Sbert, Miquel Feixas, Jaume Rigau, Miguel Chover, and Ivan Viola
2009

Introductory Tiling Theory for Computer Graphics
Craig S. Kaplan
2009

Practical Global Illumination with Irradiance Caching
Jaroslav Krivanek, Pascal Gautron
2009

Wang Tiles in Computer Graphics
Ares Lagae
2009

Virtual Crowds: Methods, Simulation, and Control
Nuria Pelechano, Jan M. Allbeck, Norman I. Badler
2008

Interactive Shape Design
Marie-Paule Cani, Takeo Igarashi, Geoff Wyvill
2008

Real-Time Massive Model Rendering
Sung-eui Yoon, Enrico Gobbetti, David Kasik, Dinesh Manocha
2008

High Dynamic Range Video
Karol Myszkowski, Rafal Mantiuk, Grzegorz Krawczyk
2008

GPU-Based Techniques for Global Illumination Effects
László Szirmay-Kalos, László Szécsi, Mateu Sbert
2008

High Dynamic Range Image Reconstruction
Asla M. Sá, Paulo Cezar Carvalho, Luiz Velho
2008

High Fidelity Haptic Rendering
Miguel A. Otaduy, Ming C. Lin
2006

A Blossoming Development of Splines
Stephen Mann
2006

Copyright © 2010 by Morgan & Claypool
All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means—electronic, mechanical, photocopy, recording, or any other except for brief quotations in printed reviews, without the prior permission of the publisher.

Rethinking Quaternions: Theory and Computation
Ron Goldman
www.morganclaypool.com

ISBN: 9781608454204 paperback

ISBN: 9781608454211 ebook

DOI: 10.2200/S00292ED1V01Y201008CGR013

A Publication in the Morgan & Claypool Publishers series

SYNTHESIS LECTURES ON COMPUTER GRAPHICS AND ANIMATION #13

Lecture #13

Series Editor: Brian A. Barsky, University of California, Berkeley

Series ISSN

ISSN 1933-8996 print
ISSN 1933-9003 electronic

Rethinking Quaternions

Theory and Computation

Ron Goldman
Rice University

SYNTHESIS LECTURES ON COMPUTER GRAPHICS AND ANIMATION #13

MORGAN & CLAYPOOL PUBLISHERS

ABSTRACT

Quaternion multiplication can be used to rotate vectors in three-dimensions. Therefore, in computer graphics, quaternions have three principal applications: to increase speed and reduce storage for calculations involving rotations, to avoid distortions arising from numerical inaccuracies caused by floating point computations with rotations, and to interpolate between two rotations for key frame animation. Yet while the formal algebra of quaternions is well-known in the graphics community, the derivations of the formulas for this algebra and the geometric principles underlying this algebra are not well understood. The goals of this monograph are

- to provide a fresh, geometric interpretation for quaternions, appropriate for contemporary computer graphics, based on mass-points;
- to present better ways to visualize quaternions, and the effect of quaternion multiplication on points and vectors in three dimensions using insights from the algebra and geometry of multiplication in the complex plane;
- to derive the formula for quaternion multiplication from first principles;
- to develop simple, intuitive proofs of the sandwiching formulas for rotation and reflection;
- to show how to apply sandwiching to compute perspective projections.

In addition to these theoretical issues, we also address some computational questions. We develop straightforward formulas for converting back and forth between quaternion and matrix representations for rotations, reflections, and perspective projections, and we discuss the relative advantages and disadvantages of the quaternion and matrix representations for these transformations. Moreover, we show how to avoid distortions due to floating point computations with rotations by using unit quaternions to represent rotations. We also derive the formula for spherical linear interpolation, and we explain how to apply this formula to interpolate between two rotations for key frame animation. Finally, we explain the role of quaternions in low-dimensional Clifford algebras, and we show how to apply the Clifford algebra for R^3 to model rotations, reflections, and perspective projections. To help the reader understand the concepts and formulas presented here, we have incorporated many exercises in order to clarify and elaborate some of the key points in the text.

KEYWORDS

complex number, mass-point, rotation, reflection, perspective projection, quaternion, sandwiching

To lovers and haters of quaternions.

And when he had apprehended him, he put him in prison, and delivered him to four quaternions of soldiers to keep him; intending after Easter to bring him forth to the people. Acts 12:4

Contents

Preface ..... **xi**

I. Theory **1**

1. Complex Numbers 3
2. A Brief History of Number Systems and Multiplication 11
   - 2.1 Multiplication in Dimensions Greater Than Two 14
3. Modeling Quaternions 17
   - 3.1 Mass-Points: A Classical Model for Contemporary Computer Graphics 17
   - 3.2 Arrows in Four Dimensions 21
   - 3.3 Mutually Orthogonal Planes in Four Dimensions 22
4. The Algebra of Quaternion Multiplication 27
5. The Geometry of Quaternion Multiplication 37
6. Affine, Semi-Affine, and Projective Transformations in Three Dimensions 47
   - 6.1 Rotation 49
   - 6.2 Mirror Image 54
   - 6.3 Perspective Projection 59
     - 6.3.1 Perspective Projection and Singular 4×4 Matrices 60
     - 6.3.2 Perspective Projection by Sandwiching with Quaternions 62
   - 6.4 Rotorperspectives and Rotoreflections 72
7. Recapitulation: Insights and Results 77

II. Computation **81**

8. Matrix Representations for Rotations, Reflections, and Perspective Projections 83
   - 8.1 Matrix Representations for Quaternion Multiplication 83
   - 8.2 Matrix Representations for Rotations 85
   - 8.3 Matrix Representations for Mirror Images 88
   - 8.4 Matrix Representations for Perspective Projections 90

9. Applications 95
9.1 Efficiency: Quaternions Versus Matrices 95
9.2 Avoiding Distortion by Renormalization 96
9.3 Key Frame Animation and Spherical Linear Interpolation 97
10. Summary—Formulas From Quaternion Algebra 101

III. Rethinking Quaternions and Clifford Algebras 107
11. Goals and Motivation 109
12. Clifford Algebras and Quaternions 111
13. Clifford Algebra for the Plane 113
14. The Standard Model of the Clifford Algebra for Three Dimensions 117
14.1 Scalars, Vectors, Bivectors, and Pseudoscalars 117
14.2 Wedge Product and Cross Product 118
14.3 Duality 119
14.4 Bivectors 121
14.5 Quaternions 122
15. Operands and Operators—Mass-Points and Quaternions 125
15.1 Odd Order: Mass-Points 125
15.2 Even Order: Quaternions 127
16. Decomposing Mass-Points Into Two Mutually Orthogonal Planes 129
16.1 Action of $q(b, \theta)$, on $b^{\perp}$ 130
16.2 Action of $q(b, \theta)$, on $b^{\parallel}$ 131
16.3 Sandwiching 134
17. Rotation, Reflection, and Perspective Projection 137
17.1 Rotation 138
17.2 Mirror Image 139
17.3 Perspective Projection 141
18. Summary 145
19. Some Simple Alternative Homogeneous Models for Computer Graphics 149

References 153

Further Reading 155

Author Biography 157

Preface

All the great ideas have been thought before; the trick is to think them again.
Goethe

Quaternions are vectors in four dimensions endowed with a rule for multiplication that is associative but not commutative, distributes through addition, contains an identity, and for which each nonzero vector in four dimensions has a unique inverse. Ever since the discovery of quaternion multiplication, quaternions have been used to rotate vectors in three dimensions by sandwiching a vector in three dimensions between a unit quaternion and its conjugate [Hamilton, 1866].

In contemporary computer graphics, quaternions have three principal applications. First, quaternions can be used to reduce storage and to speed calculations involving rotations. A quaternion is represented by just four scalars, in contrast to a 3×3 rotation matrix, which has nine scalar entries. Also, to compose two rotations with quaternion multiplication requires only 16 scalar multiplications, whereas composing two rotations with matrix multiplication uses 27 scalar multiplications (see Section 9.1). Second, quaternions can be used to avoid distortions that inevitably arise in scenes from numerical inaccuracies introduced by floating point computations involving rotations. Unlike rotation matrices, quaternions are easily renormalized, so distortions of lengths and angles can be avoided more simply by replacing rotation matrices with unit quaternions (see Section 9.2). Third, in key frame animation, we can readily interpolate smoothly between two rotations represented by unit quaternions by using spherical linear interpolation (SLERP) [Shoemake, 1985] (see also Section 9.3). In contrast, it is not so simple to interpolate smoothly between two rotation matrices. Other potential applications of quaternions in computer graphics include practical methods for tubing and texturing smooth curves and surfaces using optimal orthonormal frames [Hanson, 2006], better ways to visualize streamlines [Hanson, 2006], and effective techniques for generating and analyzing three-dimensional Pythagorean hodograph curves [Farouki, 2008].

Yet although the algebra of quaternions—multiplication, sandwiching, interpolation—is well established in computer graphics [Foley *et al.*, 1990], the underlying geometry of quaternions is not well understood. The formulas for multiplication and sandwiching work, but it is hard to see how anyone ever came up with these formulas. Even the geometric meaning of a quaternion—a scalar

added to a vector—is an enigma. The purpose of this book is to develop a better intuitive geometric understanding of quaternions, as well as to remove much of the mystery surrounding quaternion algebra.

The goal of this monograph is to make five principal theoretical contributions:

1. To provide a fresh, geometric interpretation for quaternions, appropriate to contemporary computer graphics, by invoking mass-points. Thus this monograph is a sequel to [Goldman, 2002], extending multiplication to the space of mass-points;
2. To present better ways to visualize quaternions and the effect of quaternion multiplication on points and vectors in three dimensions based on insights from the algebra and geometry of multiplication in the complex plane;
3. To derive the formula for quaternion multiplication from first principles;
4. To develop simple, intuitive proofs of the sandwiching formulas for rotation and reflection;
5. To show how to apply sandwiching to compute perspective projections.

In addition to these theoretical issues, we shall also address some computational questions. Our objectives here are to:

1. Develop straightforward explicit formulas for converting between quaternion and 4×4 matrix representations for rotations, reflections, and perspective projections;
2. Discuss the relative merits of the quaternion and matrix representations for these three transformations;
3. Explain how to avoid distortions due to floating point computations with rotations by using unit quaternions to represent rotations;
4. Derive the formula for spherical linear interpolation (SLERP) and explain how to apply this formula to interpolate between two rotations for key frame animation.

Recently several authors have suggested Clifford algebras as the appropriate framework for the study of contemporary computer graphics [Dorst *et al.*, 2007]. Quaternions form a subalgebra of the Clifford algebra for R^3. Therefore, in Part III of this monograph, we investigate the role of quaternions in the Clifford algebra for R^3. Here our purpose is to:

1. Provide a straightforward explanation of Clifford algebra for the uninitiated;
2. Present a simpler model of Clifford algebra for the study of computer graphics than the one proposed by current proponents of Clifford algebra;
3. Explain the role of quaternions in the Clifford algebra for R^3 and relate this role back to what we have already learned in about quaternions in Part I of this text.

To facilitate these goals, this monograph is divided into three parts: Part I deals with the theory of quaternion multiplication; Part II is devoted to computational issues; Part III investigates the role of quaternions in the Clifford algebra for R^3. To help the reader better understand the concepts, insights, and formulas presented here, we have incorporated many practice exercises throughout the text.

We begin in Chapter 1 with a brief review of complex numbers; these well-known lower-dimensional analogues of quaternions provide the natural foundation for understanding quaternions. But before we launch into our study of quaternions using insights from complex numbers, we provide in Chapter 2 a brief overview of the meaning of multiplication for different number systems—the natural numbers, the integers (positive and negative), the rational numbers, the real numbers, and the complex numbers—each system generated by solving more and more complicated polynomial equations. We observe that after the complex numbers no new numbers are generated by solving more complicated polynomial equations, so the notion of numbers in higher dimensions must take on new meaning. We find this meaning in geometry rather than algebra, in representing rotations in higher dimensions rather than embodying novel solutions to polynomial equations. This interpretation leads us naturally to quaternions.

In Chapter 3, we introduce three geometric models for thinking about quaternions: mass-points in three dimensions, vectors in four dimensions, and pairs of mutually orthogonal planes in four dimensions, one isomorphic to the complex plane and the other isomorphic to a quaternion multiple of the complex plane. Chapter 4 is devoted to investigating the algebra of quaternion multiplication. Here we derive the formula for the product of two quaternions from first principles: the associative and distributive laws for multiplication and the rule that the length of the product of two quaternions must be equal to the product of their lengths. In Chapter 5, we provide a fresh, intuitive, geometric interpretation of quaternion multiplication as rotation in two mutually orthogonal planes in four dimensions: one plane isomorphic to the complex numbers where multiplication on the left and the right rotates vectors by the same amount in the same direction and a second plane isomorphic to a quaternion multiple of the complex numbers, where multiplication on the left and the right rotates vectors by the same amount in opposite directions. Chapter 6 is devoted to using this simple, geometric interpretation of quaternion multiplication to develop straightforward, intuitive derivations of the sandwiching formulas for rotation, mirror image, and perspective projection in three dimensions. While the sandwiching formulas for rotation and reflection are well-known, the sandwiching formula for perspective projection in Section 6.3 is new and does not appear in the work of previous writers on quaternions. Here the interpretation of quaternions as mass-points plays a key role. In Section 6.4, we also provide straightforward geometric interpretations for how left and right multiplication by quaternions as well as some more general sandwiching formulas affects vectors in three dimensions. We recapitulate our main conceptual insights and principal

theoretical results in Chapter 7. This summary ends Part I of this monograph on the theory of quaternions.

In Part II we turn our attention from theoretical to computational issues. In Chapter 8 we develop straightforward explicit formulas for converting back and forth between quaternion and matrix representations for rotations, reflections, and perspective projections. Chapter 9 is devoted to applications of quaternions in computer graphics. In Section 9.1 we discuss the relative advantages and disadvantages of the quaternion and matrix representations for rotations, reflections, and perspective projections. Section 9.2 explains how to avoid distortions due to floating point computations with rotations by using unit quaternions to represent rotations. In Section 9.3 we derive the formula for spherical linear interpolation (SLERP), and we explain how to apply this formula to interpolate between two rotations for key frame animation. We close Part II in Chapter 10 with a brief summary of the main formulas presented in the body of this text.

Part III is devoted to understanding the role of quaternions in Clifford algebra. We begin in Chapter 11 with a general overview of our goals and motivation. Chapter 12 contains a brief description of the general Clifford algebras for n-dimensional vector spaces. Here we observe that for $n \geq 2$ each of these Clifford algebras (with signature n) contains the quaternions as a subalgebra. The Clifford algebra for $\boldsymbol{R}^2$ is actually isomorphic to the quaternions, so in Chapter 13 we investigate the Clifford algebra for $\boldsymbol{R}^2$ and show how this Clifford algebra is used to study conformal (i.e., angle preserving) linear maps on $\boldsymbol{R}^2$. In Chapter 14 we provide a brief review for the uninitiated of the standard objects—scalars, vectors, bivectors, and pseudoscalars—along with some of the basic formulas—Clifford product, wedge product, duality—for the Clifford algebra of R^3. Readers already familiar with Clifford algebra can skip this tutorial. In Chapter 15 we introduce the operators and operands—quaternions and mass-points—by exploiting the pseudoscalars to represent mass. Chapter 16 is devoted to studying the action of the unit quaternions on the space of mass-points, and Chapter 17 shows how to apply the sandwiching operators to compute rotation, mirror image, and perspective projection on points as well as on vectors in $\boldsymbol{R}^3$. Many of these results will already be known to readers familiar with Clifford algebra, but the material on perspective projection in Section 17.3 appears once again to be completely new. We summarize our principal insights and results in Chapter 18. Finally, we close in Chapter 19 with a critical assessment of our approach to the Clifford algebra of $\boldsymbol{R}^3$, and we provide here some simple, alternative approaches to a homogeneous model for contemporary computer graphics.

There are two fundamental insights in this text: one algebraic and the other geometric. The algebra is covered in Chapter 4, the geometry in Chapter 5. The rest of the text either provides preliminary results leading up to these two chapters or contains consequences of the results presented in these two chapters.

The key algebraic insight is that the property that the length of the product is equal to the product of the lengths along with the standard associative, distributive, and identity axioms for multiplication are enough to completely characterize the formula for the product of two quaternions. This insight goes back to Hurwitz [1898], who used this idea to prove algebraically that the only real vector spaces with multiplication and division are in dimensions 1, 2, and 4, corresponding to the real numbers, the complex numbers, and the quaternions [see also Conway and Smith, 2003]. (A nonassociative rule for multiplication exists in dimension 8; these numbers are called the *octonions*—see again Conway and Smith, 2003.) We shall show in Chapter 4 how to apply this approach in a more geometric fashion to derive similar results.

The second key insight is geometric. Multiplication by a unit quaternion represents a double isoclinic rotation in four dimensions [Mebius, 2005]—that is, there are two orthogonal planes in four dimensions where vectors are rotated by the same angle. For left multiplication, the rotations in both planes are counterclockwise, but for right multiplication, the rotation in one plane is counterclockwise, while the rotation in the other plane is clockwise; thus left and right quaternion multiplication generate left and right screws in $\boldsymbol{R}^4$ [Du Val, 1964]. This observation allows us to apply sandwiching to combine two rotations in order to cancel the rotation in one of these planes and thereby generate a simple rotation in four dimensions. This insight is what ultimately allows us in Chapter 6 to apply sandwiching with unit quaternions to model rotations, reflections, and perspective projections in three dimensions. Hence these three basic transformations in three dimensions are modeled by simple rotations in four dimensions. Therefore the point of view that we shall adopt here is that the fundamental transformations are precisely the simple rotations in four dimensions.

Thus the main ideas and insights in this monograph are not new but are rather based on the work of many predecessors. I have simply tried to update, organize, and apply these seminal ideas in ways that can be more easily understood by contemporary practitioners of computer graphics.

Only elementary matrix algebra along a passing familiarity with complex numbers is required background for reading this work. I have tried, however, whenever possible to use coordinate free methods. Therefore the reader should also have a rudimentary knowledge of vector algebra, especially of dot and cross product. Minimally the reader should know that

$$u \cdot v = 0 \Leftrightarrow u \perp v$$

$$u \times v \perp u, v.$$

Most of the technical terms used throughout this text are standard in computer graphics and will be familiar even to novice readers. Two important concepts that may be less familiar are expressed by the terms *conformal map* and *isometry*. A transformation is said to be a *conformal map*

if it preserves angles; a transformation is said to be an *isometry* if it preserves lengths. All isometries are conformal maps, but not all conformal maps are isometries. For example, rotation is an isometry; uniform scaling is not an isometry, but uniform scaling is a conformal map. Armed with these basic concepts, the reader should now be fully ready to proceed to the text.

I would like thank Rida Farouki for reigniting my interest in quaternions by explaining to me that multiplying by a unit quaternion rotates quaternions in four dimensions. One of the main themes of this monograph is that the simplest way to understand how quaternion multiplication generates different transformations in three dimensions is to understand how quaternion multiplication induces simple rotations in four dimensions—that is, rotations in a single plane in four dimensions. I would also like to thank Joe Warren for encouraging me to develop a more intuitive approach to understanding quaternions, as well as for reading several initial drafts of Part I of this manuscript and suggesting better ways to organize and present some of this material. In addition, I would like to thank Andrew Hanson for some lively discussions along with several insightful observations concerning the ideas presented in Part I. I am indebted to Leo Dorst and Steve Mann for reading a preliminary draft of Part III and providing valuable comments, criticisms, and suggestions. Part III of the text is much improved as a result of their observations. Finally, Rida Farouki and Andrew Hanson read an initial draft of this entire manuscript and suggested many worthwhile improvements which I have tried to incorporate into the body of this monograph.

Most of the material in Part I first appeared in my *Graphical Models* paper "Understanding Quaternions," and much of the material in Part II is taken from Chapter 15 of my book, "An Integrated Introduction to Computer Graphics and Geometric Modeling." I would like to thank Elsevier and Taylor & Francis for allowing me to reuse this material in this extended opus.

PART I

Theory

CHAPTER 1

Complex Numbers

Many of the properties of quaternions can be understood as extensions of properties of the complex numbers. Therefore we begin our study of quaternions with a brief review of complex numbers and complex multiplication.

Complex numbers are often represented by vectors in two dimensions. The complex number $w = a + bi$ corresponds to the vector $v = (a, b)$ in the xy-plane. Addition, subtraction, and scalar multiplication of complex numbers correspond to coordinate-free operations on vectors in the plane. Rectangular coordinates are introduced in practice to perform these computations, but in principle there is no need for preferred directions—for coordinate axes—to define addition, subtraction, or scalar multiplication for vectors in the plane (see Figure 1.1).

Complex multiplication is different. For complex multiplication, we must choose a preferred direction: the direction of the identity vector for multiplication. We denote this identity vector by 1 and we associate this vector with the unit vector along the positive x-axis. The unit vector perpendicular to the identity vector we denote by i and we associate this vector with the unit vector along the positive y-axis. Relative to this coordinate system for every vector v in the plane there are constants a, b such that

$$v = a1 + bi.$$

Typically, we drop the symbol 1, and we write simply

$$v = a + bi.$$

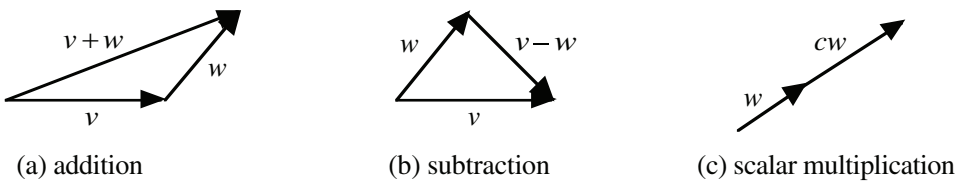


FIGURE 1.1: Coordinate-free geometric constructions for (a) addition, (b) subtraction, and (c) scalar multiplication for vectors.

In the complex number system, i denotes $\sqrt{-1}$, so

$$i^2 = -1. \tag{1.1}$$

Since 1 is the identity for multiplication, we have the rules

$$(1)(1) = 1, \ (i)(1) = (1)(i) = i, \quad (i)(i) = -1. \tag{1.2}$$

Therefore, if we want complex multiplication to distribute through addition, then we must define

$$(a + b\,i)(c + d\,i) = a(c + d\,i) + bi(c + d\,i) = (ac - bd) + (ad + bc)i. \tag{1.3}$$

Using Equation (1.3), it is straightforward to verify that complex multiplication is associative, commutative, and distributes through addition (see Exercise 1.1).

Every nonzero complex number has a multiplicative inverse. Let $w = a + bi$. The complex conjugate of w, denoted by w^*, is defined by

$$w^* = a - bi. \tag{1.4}$$

By Equation (1.3),

$$w\,w^* = a^2 + b^2 = |w|^2, \tag{1.5}$$

where $|\,w\,|$ denotes the length of the vector $w = (a, b)$. Therefore,

$$w^{-1} = \frac{w^*}{|w|^2}.$$

From Equations (1.4) and (1.5), it is easy to verify that

$$(w_1 w_2)^* = w_1^* w_2^*. \tag{1.6}$$

Hence by Equations (1.5) and (1.6)

$$|w_1 w_2|^2 = (w_1 w_2)(w_1 w_2)^* = (w_1 w_1^*)(w_2 w_2^*) = |w_1|^2\,|w_2|^2,$$

so

$$|w_1 w_2| = |w_1|\,|w_2|. \tag{1.7}$$

(For an alternative derivation of Equation (1.7), see Exercise 1.6.)

Equation (1.3) encapsulates the algebra of complex multiplication, but what is the geometric meaning of complex multiplication? Figure 1.1 illustrates the geometry underlying addition, subtraction, and scalar multiplication for complex numbers. To understand the geometry underlying

complex multiplication, observe that multiplication by a fixed complex number w is a linear transformation on vectors in the plane, since by the distributive property

$$w(z_1 + z_2) = wz_1 + wz_2.$$

Moreover, multiplication by a complex number of unit length is an isometry (preserves length) because by Equation (1.7)

$$|w_1| = 1 \Rightarrow |w_1 w_2| = |w_2|.$$

But the only linear isometries in the plane are rotations, so multiplication by a complex number of length one must rotate vectors in the plane.

Let's look at some examples. Consider first multiplication with i. If $v = a + bi$, then

$$iv = i(a + bi) = -b + ai.$$

Thus if $v = (a,b)$, then $iv = (-b,a)$, so

$$v \cdot iv = (a,b) \cdot (-b,a) = 0.$$

Hence iv is perpendicular to v. Therefore multiplication by i is equivalent to rotation by 90°. Also,

$$(-1)(v) = -(a + b\,i) = -a - b\,i = -v,$$

so multiplication by -1 is equivalent to rotation by 180°. Thus there is indeed a link between rotation and complex multiplication. Let's explore this connection further.

Suppose that we want to rotate a vector w by the angle θ. Let $w^{\perp}$ denote the vector orthogonal to w of the same length as w. Then, after rotation,

$$w_{new} = \cos(\theta)w + \sin(\theta)w^{\perp} \quad \text{(see Figure 1.2).}$$

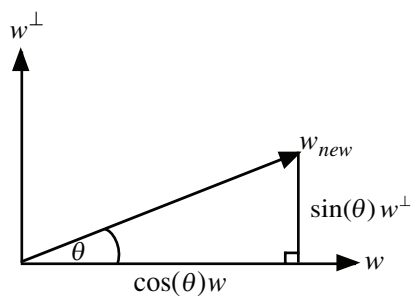


FIGURE 1.2: Rotating the vector w by the angle θ.

But if $w = (u, v)$, then $w^{\perp} = (-v, u)$, so

$$u_{new} = u\cos(\theta) - v\sin(\theta)$$
$$v_{new} = v\cos(\theta) + u\sin(\theta).$$

Now let $z(\theta)$ be the unit vector that makes an angle θ with the x-axis (see Figure 1.3). Then

$$z(\theta) = \cos(\theta) + i\sin(\theta),$$

so

$$z(\theta)w = (\cos(\theta) + i\sin(\theta))(u + iv) = \left(\underbrace{u\cos(\theta) - v\sin(\theta)}_{u_{new}}\right) + \left(\underbrace{u\sin(\theta) + v\cos(\theta)}_{v_{new}}\right) i$$

Thus multiplying the complex number w by $z(\theta)$ is equivalent to rotating the vector w by the angle θ. Moreover, since complex multiplication is associative, we can compose two rotations simply by multiplying the associated complex numbers—that is, for all complex numbers w

$$z(\phi + \theta)\, w = z(\phi)z(\theta)w,$$

so

$$z(\phi + \theta) = z(\phi)z(\theta).$$

Rotation in the plane closely resembles exponentiation because composing two rotations is equivalent to adding the corresponding angles of rotation. Since complex multiplication represents

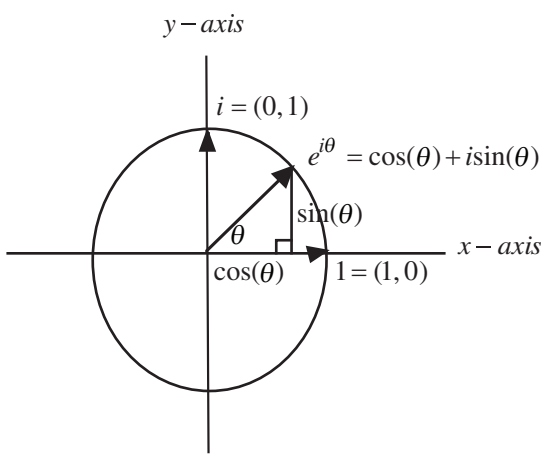


FIGURE 1.3: Multiplying a complex number w by the unit vector $z(\theta) = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ is equivalent to rotating the vector w by the angle θ.

rotation, complex multiplication is also closely related to exponentiation. Let z be an arbitrary complex number. Define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

Then

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \cdots .$$

But $i^2 = -1$, so substituting and collecting terms, we arrive at the following fundamental identity:

Euler's formula

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos(\theta) + i\sin(\theta) . \tag{1.8}$$

Since

$$z(\theta) = \cos(\theta) + i\sin(\theta) = e^{i\theta},$$

we see once again that

$$z(\phi)z(\theta) = e^{i\phi}e^{i\theta} = e^{i(\phi+\theta)} = z(\phi + \theta),$$

so multiplying two complex numbers of unit length is equivalent to adding the angles that these vectors form with the x-axis.

What happens if we multiply a fixed complex number w by an arbitrary complex number z? If $z = r$, where r is a real number, then $z\,w = r\,w$, so the effect is to scale the vector w by the constant r. Thus, both rotation and scaling can be represented by complex multiplication. Finally, if $z = x + i\,y$ is an arbitrary complex number, then we can write z in polar form as

$$z = re^{i\theta} = rz(\theta)$$

where $r = |z|$ and $\theta = arctan(y / x)$. Hence

$$zw = rz(\theta)w ,$$

so the effect of multiplying w by z is to rotate w by the angle θ and to scale the result by the constant r. Thus any conformal (angle preserving) linear transformation of vectors in the plane can be represented by multiplication with a single complex number.

Exercises:

1.1. Using Equation (1.3), verify that complex multiplication is associative, commutative, and distributes through addition.

1.2. Let (r,θ) denote the polar coordinates of a point in the plane. Show that in polar coordinates complex multiplication is equivalent to

$$(r_1,\theta_1)\,(r_2,\theta_2) = (r_1 r_2,\, \theta_1 + \theta_2).$$

1.3. Here we shall derive the half angle formulas used throughout the text.

a. Using Euler's formula—Equation (1.8)—show that

i. $\cos(2\phi) + i\sin(2\phi) = (\cos(\phi) + i\sin(\phi))^2$.

b. Conclude from part *a* that

i. $\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi) = 2\cos^2(\phi) - 1$

ii. $\sin(2\phi) = 2\sin(\phi)\cos(\phi)$.

c. Conclude from part *b* that

i. $\cos(\phi/2) = \sqrt{\dfrac{1+\cos(\phi)}{2}}$

ii. $\sin(\phi/2) = \sqrt{\dfrac{1-\cos(\phi)}{2}}$.

1.4. Let $z = (x,y) = x + iy$ and $w = (u,v) = u + iv$ be arbitrary vectors in the plane, and let θ be the angle between z and w. Show that

a. $zw^* = z \cdot w + i\det(z,w)$

b. $zw^* = |z|\,|w|\cos(\theta) \pm i\,area(z,w)$.

1.5. Let w_1, w_2, w be complex numbers and let $p(z) = a_n z^n + \ldots + a_0$ be a polynomial in the complex variable z with real coefficients $a_0, \ldots, a_n$.

a. Show that

i. $(w_1 + w_2)^* = w_1^* + w_2^*$

ii. $(w_1 w_2)^* = w_1^* w_2^*$

iii. $(w^n)^* = (w^*)^n$

b. Conclude from part *a* that if $p(w) = 0$, then $p(w^*) = 0$.

1.6. Let a,b,x,y be real numbers, and let $z_1 = a + bi$, $z_2 = x + yi$ be complex numbers.

a. Verify Brahmagupta's identity:

$(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (ay + bx)^2$.

b. Using Brahmagupta's identity, prove that
$|z_1 z_2| = |z_1||z_2|$.

1.7. Let a, b, c and A, B, C be the complex numbers that represent the vertices of two triangles S, T in the complex plane. Show that S, T are similar triangles if and only if

$$\frac{b-a}{c-a} = \frac{B-A}{C-A}.$$

CHAPTER 2

A Brief History of Number Systems and Multiplication

Before we launch into our study of quaternions, we are going to review briefly multiplication for standard number systems to see what we should expect for multiplication in higher dimensions.

The natural numbers $\boldsymbol{N} = \{0, 1, 2, \ldots\}$ are the numbers we use for counting. For these numbers, multiplication is simply repeated addition—that is, for $m, n \in \boldsymbol{N}$

$$m\,n = \underbrace{n + \cdots + n}_{m\ times}.$$

For the natural numbers, multiplication is associative, commutative, and distributes through addition. We can understand these three properties as natural symmetries by invoking simple geometric figures to model these algebraic axioms (see Figures 2.1 and 2.2).

The associative property—$m \times (n \times p) = (m \times n) \times p$—can also be modeled geometrically by stacking the dots in a 3-D array of order $m \times n \times p$ and then rotating the dots by 90° in three dimensions.

One reason we need to extend the natural numbers $\boldsymbol{N}$ to ever larger number systems is to introduce solutions to more and more complicated polynomial equations.

i. $n\,x = m \rightarrow$ rational numbers

ii. $x + n = 0 \rightarrow$ negative numbers

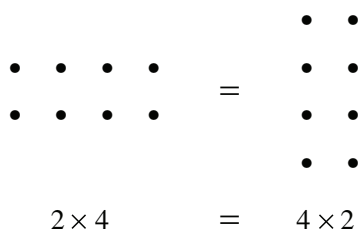
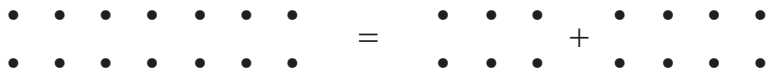


FIGURE 2.1: The commutative property for multiplication of natural numbers: Rotating the dots by 90° generates a symmetric figure, which does not change the total number of dots.



$$2 \times (3+4) \quad = \quad 2 \times 3 \quad + \quad 2 \times 4$$

FIGURE 2.2: The distributive property for multiplication of natural numbers: Splitting or concatenating the dots does not change the total number of dots.

iii. $x^2 - 2 = 0 \rightarrow$ real numbers
iv. $x^2 + 1 = 0 \rightarrow$ complex numbers

For the rational numbers, we again define multiplication by natural numbers using repeated addition:

$$m\left(\frac{p}{q}\right) = \underbrace{\frac{p}{q} + \cdots + \frac{p}{q}}_{m \text{ times}}.$$

We also require that multiplication of rational numbers is associative, commutative, and distributes through addition. It follows easily from these rules that

$$\frac{p}{q}\frac{r}{s} = \frac{pr}{qs},$$

since multiplying both sides by qs will yield the same result (see Exercise 3.1).

For the negative numbers, we once again define multiplication by natural numbers using repeated addition:

$$m(-n) = \underbrace{-n + \cdots + -n}_{m \text{ times}}.$$

Once again we require that multiplication of negative numbers must be associative, commutative, and distribute through addition. But now we need a new rule. Since +1 is the identity for multiplication, we know that *plus* × *plus* = *plus* and *plus* × *minus* = *minus* × *plus* = *minus*, but what is the rule for *minus* × *minus*? The new rule is that *minus* × *minus* = *plus* because by the distributive property

$$0 = (-1)(1-1) = -1(1) + (-1)(-1) \Rightarrow (-1)(-1) = 1.$$

Notice that multiplication by negative numbers preserves length, since

$$|m\,n| = |m|\,|n|.$$

Some real numbers like $\sqrt{2}$ are solutions of simple polynomial equations, but other real numbers like π are not the solution of any polynomial equation. Nevertheless, every real number is the limit of a sequence of (positive or negative) rational numbers, so we simply extend the rules for multiplication—associative, commutative, and distributive—from the rationals to the reals.

Finally for the complex numbers, we also define multiplication by natural numbers using repeated addition:

$$m\,w = \underbrace{w + \cdots + w}_{m\ times}.$$

Moreover, by definition

$$i^2 = -1.$$

If we require that complex multiplication must also distribute through addition, then as we have seen in Chapter 1, we arrive at the following general rule for the product of two complex numbers:

$$(a + bi)(c + di) = a(c + di) + bi(c + di) = (ac - bd) + (ad + bc)i.$$

Now it is straightforward to check that with this rule complex multiplication is indeed associative, commutative, and distributes through addition. Moreover, complex multiplication also preserves length. Indeed if we define

$$|a + b\,i| = \sqrt{a^2 + b^2} = \text{length of the vector } (a,b),$$

then, again as we have seen in Chapter 1, it is easy to verify that for any two complex numbers w, z

$$|\,w\,z\,| = |\,w\,|\;|\,z\,|.$$

We have extended our notion of numbers from the natural numbers to the rational, negative, real, and complex numbers by solving more and more complicated polynomial equations. But here we must stop because of the following theorem.

Theorem 2.1: *Every polynomial of degree n has n complex roots (counting multiplicity).*

Now if r is a root of a polynomial $P(t)$, then it is easy to verify that $t - r$ is a factor of $P(t)$ (see Exercise 2.2). Hence we cannot introduce any new numbers by solving additional polynomial equations, since a polynomial of degree n can have at most n linear factors. Thus, by Theorem 2.1, all the roots of $P(t)$ are complex numbers. Even the equation $x^2 + i = 0$ has two complex solutions, since it is easy to verify that

$$(\pm\sqrt{2}/2 \pm i\sqrt{2}/2)^2 = i.$$

Therefore our trick for extending multiplication from one dimension (real numbers) to two dimensions (complex numbers) cannot be extended to higher dimensions. What then might multiplication mean in dimensions greater than two?

2.1 MULTIPLICATION IN DIMENSIONS GREATER THAN TWO

Table 2.1 provides another, more geometric interpretation of multiplication in one and two dimensions that we have already encountered which is independent of the algebra of polynomial equations.

TABLE 2.1: Geometric interpretation of multiplication in dimensions 1 and 2

REAL MULTIPLICATION	COMPLEX MULTIPLICATION
• 1 represents the identity	• 1 represents the identity
• –1 represents rotation by 180°	• –1 represents rotation by 180°
• $c > 0$ represents scaling	• i represents rotation by 90°
	• $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ represents rotation by θ
	• $c > 0$ represents scaling

Therefore what we might hope to model using multiplication in higher dimensions is rotation and scaling—that is, conformal (i.e., angle preserving) linear transformations. In particular, we would like multiplication in higher dimensions to have the following properties:

In analogy with real and complex multiplication, we would also like multiplication in higher dimensions to have the usual symmetry properties: to be associative and to distribute through addition. Notice that the distributive property is particularly important, since if multiplication distributes through addition, then multiplication is a linear map and hence can be modeled by matrix multiplication. But if we want multiplication to model rotation, then we cannot expect multiplication in higher dimensions to be commutative, since even in three dimensions rotation is not commutative. (For example, rotating a vector parallel to the x-axis first around the x-axis and then around the z-axis is not equivalent to rotating the same vector first around the z-axis and then around the x-axis. In the first case, the rotated vector lies in the xz-plane; in the second case, the vector is rotated out of the xz-plane.) Indeed matrix multiplication, which can be used to represent rotations in higher dimensions, is not commutative. Notice too that we are required to choose a direction for the multi-

TABLE 2.2: Multiplication as Rotation and Scaling

MULTIPLICATION AS ROTATION	MULTIPLICATION AS SCALING
• 1 Represents the identity	• Choose a direction—not coordinate free
• Unit vectors represent rotations	•Similar to complex multiplication
• $c > 0$ represents scaling	• Similar to real multiplication
RULES	REASONS
• Associative	•Rotation is associative
•Distributes through addition	• Rotation is a linear transformation
• Identity and inverses	• Rotations can be undone
• Not Commutative	• Rotation in 3-D not commutative
• Preserves length ($\|pq\| = \|p\|\,\|q\|$)	• Rotation is an isometry

plicative identity 1, so multiplication in higher dimensions, like multiplication in the complex plane, cannot be completely coordinate free.

Finally, we shall insist that multiplication should preserve length—that is,

$$\bullet \; \| p\, q \| = \| p \| \, \| q \| \tag{2.1}$$

Rotation is an isometry. We are going to use scalar multiplication to model uniform scaling, so we expect to use unit vectors to represent rotations. Thus, as with the real and complex numbers, the length of the product should be equal to the product of the lengths. Equation (2.1) is a crucial property; we shall make extensive use of this property in Chapter 4.

We shall see in Chapter 4 that there is no rule for multiplication in three dimensions that satisfies these constraints, so we turn our attention next to multiplication in four dimensions.

Exercises:

2.1. Use the associative and commutative rules to show that multiplying $\frac{p}{q}\frac{r}{s}$ and $\frac{pr}{qs}$ by qs yields the same result. Conclude that $\frac{p}{q}\frac{r}{s} = \frac{pr}{qs}$.

2.2. Show that if r is a root of a polynomial $P(t)$, then $t - r$ is a factor of $P(t)$.

(Hint: Consider $P(t) - P(r)$.)

2.3. Show that $(t - r)^d$ is a factor of a polynomial $P(t)$ if and only if $P(r) = P'(r) = \ldots = P^{(d-1)}(r) = 0$.

(Hint: Consider the Taylor expansion of $P(t)$ at r.)

• • • •

CHAPTER 3

Modeling Quaternions

Complex numbers are vectors in two dimensions; quaternions are vectors in four dimensions. Modeling complex numbers by vectors in the plane makes it simple to visualize complex numbers and easy to understand the rotations induced by complex multiplication. To understand quaternions, we also need some mental images to help us both to model the vectors in four dimensions and to grasp the geometric operations on these 4-dimensional vectors induced by quaternion multiplication. Here we shall provide three such models: mass-points in three dimensions, vectors (arrows) in four dimensions, and a pair of mutually orthogonal planes in four dimensions.

As a vector space over the real numbers:

$R^4 \cong R^3 \oplus R$	(mass-points)
$R^4 \cong R^2 \oplus R^2 \cong C \oplus Cv$	(pair of orthogonal planes; here v is a unit vector in the plane in R^4 perpendicular to C),

so each of these mental models corresponds to a different way of decomposing R^4. The mass-points model ties the 4-dimensional geometry of quaternions back to the 3-dimensional geometry of the visual world. We shall see in Chapter 6 that this paradigm is crucial for understanding how sandwiching with quaternions can effectively model transformations such as rotations, reflections, and most especially perspective projections on points and vectors in three dimensions. The pair of planes model links the algebra of quaternions to the algebra of complex numbers. We shall see in Chapter 5 that this pair of planes model combined with complex multiplication is particularly powerful for visualizing the geometric effects of quaternion multiplication.

3.1 MASS-POINTS: A CLASSICAL MODEL FOR CONTEMPORARY COMPUTER GRAPHICS

Although the visual world is 3-dimensional, contemporary computer graphics typically uses four coordinates to represent points and vectors and 4×4 matrices to represent the transformations in the graphics pipeline. Four coordinates and 4×4 matrices are necessary to represent perspective projection using matrix multiplication because perspective projection is not a linear transformation in three dimensions; rather perspective projection is a rational linear (i.e., a projective) transformation.

A fourth coordinate is therefore required to represent the denominators introduced by perspective projections. If $w \neq 0$, then the four coordinates (x, y, z, w) represent the point in three dimensions located at $(x/w, y/w, z/w)$. Thus if we want to work in a vector space, the natural domain for contemporary computer graphics is not R^3, but rather R^4. Three of the dimensions are indeed spatial; the fourth dimension is . . . well what exactly is this fourth dimension?

The simplest geometric objects in three dimensions are points and vectors. Points have position, but no direction or length; vectors have direction and length, but no fixed position. Points are typically represented by dots; vectors are usually depicted by arrows. Vectors can be added, subtracted, and multiplied by scalars using the usual triangle rules for combining arrows (see Figure 1.1). But the only simple geometric operations we can perform on points are to add a vector to a point to generate a new point by translation or to subtract a point from a point to form the vector joining the two points (see Figure 3.1).

Vectors are closed under linear combinations, but points are closed only under affine combinations—combinations where the coefficients sum to 1. Indeed affine combinations can be rewritten as the sum of a point and a vector, since if the coefficients sum to 1:

$$\sum_{k=0}^{n} c_k P_k = \left(\sum_{k=0}^{n} c_k \right) P_0 + \sum_{k=1}^{n} c_k (P_k - P_0) = P_0 + \sum_{k=1}^{n} c_k (P_k - P_0) \quad .$$

Therefore, the vectors form a vector space, but the points form only an affine space. To overcome this asymmetry and to combine points and vectors into a one comprehensive geometric space, we introduce the notion of mass-points.

A *mass-point* is a point P in affine space together with a nonzero mass m. Rather than writing the pair (P, m), we shall see shortly that it is more convenient to denote a mass-point by the pair

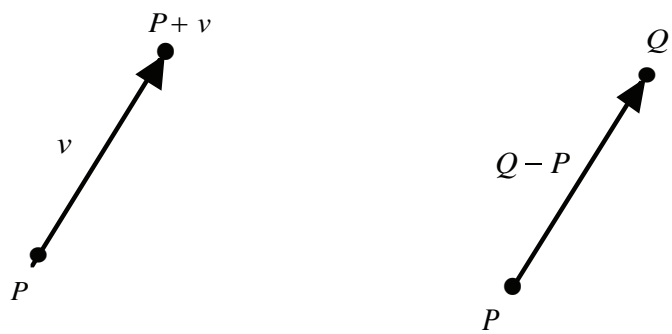


FIGURE 3.1: Adding a vector to a point and subtracting a point from a point. Notice that $P + (Q - P) = Q$, so the usual cancellation law of addition applies.

(mP, m). Here m is the mass and $P = mP/m$ is the point; the expression mP by itself has no intrinsic geometric meaning. Notice that the mass m may be negative; nevertheless we still use the term *mass-point*, even for points with negative mass. Vectors reside in this space as objects with zero mass, so for vectors v we write $(v, 0)$.

The space of mass-points and vectors forms a 4-dimensional vector space. Addition and scalar multiplication are defined component-wise; thus,

$$(m_1P_1, m_1) + (m_2P_2, m_2) = (m_1P_1 + m_2P_2, m_1 + m_2) \tag{3.1}$$

$$(-mP_1, -m) + (mP_2, m) = (m(P_2 - P_1), 0) \tag{3.2}$$

$$c\,(mP, m) = (cmP, cm) \tag{3.3}$$

$$(v, 0) + (w, 0) = (v + w, 0) \tag{3.4}$$

$$c\,(v, 0) = (cv, 0) \tag{3.5}$$

$$(mP, m) + (v, 0) = (mP + v, m) \tag{3.6}$$

Hence to add, we simply add the components of each pair; similarly, to multiply by a scalar, we multiply the components of each pair by the scalar.

Equations (3.1)–(3.6) have physical interpretations. Equation (3.1) says that the sum of two mass-points is the mass-point whose mass is the sum of the two masses located at their center of mass (see Figure 3.2). To verify this claim, we simply apply Archimedes' law of the lever, which says that two masses balance at their center of mass. Let

$$d_1 = dist\left(\frac{m_1P_1 + m_2P_2}{m_1 + m_2}, P_1\right) = \frac{m_2|P_2 - P_1|}{m_1 + m_2}$$

$$d_2 = dist\left(\frac{m_1P_1 + m_2P_2}{m_1 + m_2}, P_2\right) = \frac{m_1|P_1 - P_2|}{m_1 + m_2}.$$

FIGURE 3.2: The sum of two mass-points with masses M and m is the mass-point whose mass is the sum $M + m$ of the two masses located at their center of mass denoted here by Δ.

Then it is easy to see that

$$m_1 d_1 = m_2 d_2.$$

Thus by Archimedes' law of the lever $(m_1P_1, m_1) + (m_2P_2, m_2)$ is located at the center of mass of (m_1P_1, m_1) and (m_2P_2, m_2). If $m_1 = -m_2$, then the sum of the masses is zero. In this case, Equation (3.2) says that the sum of two mass-points $(-mP_1, -m)$ and (mP_2, m) with equal and opposite masses is the vector from P_1 to P_2 scaled by the mass m.

To multiply a mass-point (mP, m) by a scalar c, Equation (3.3) says that we multiply the mass by c and leave the position of the point unchanged.

Addition and scalar multiplication have already been defined for vectors in three dimensions (see Figure 1.1). Equations (3.4) and (3.5) say that we just carry over these definitions in the obvious manner to the space of mass-points. Thus, the 3-dimensional vectors form a subspace of the 4-dimensional vector space of mass-points.

To complete the algebra of mass-points, we need to define how to add a vector v to a mass-point (mP, m). We can imagine that the vectors carry momentum and try to visualize what happens when momentum is transferred to a mass-point. Since momentum is conserved, the larger the mass, the smaller the velocity imparted to the mass-point. In fact, since momentum is conserved, the velocity varies inversely with the mass, so the velocity imparted to the mass-point (mP, m) by the momentum vector v is simply v/m. Thus, in unit time, the mass-point at P is relocated to the new position $P + v/m$. Therefore, since the mass-point $(mP + v, m)$ is located at the affine point $P + v/m$, we define

$$(mP, m) + (v, 0) = (mP + v, m).$$

Thus, once again, to compute the sum, we simply add the components. Notice that if a unit mass is located at P, then the momentum vector v moves the mass-point $(P, 1)$ to the location $P + v$, which is the location of the standard sum of a point and a vector in affine space.

It is easy to check that with these definitions scalar multiplication distributes through addition, so the space of mass-points is indeed a vector space. The space of mass-points is 4-dimensional: three of the dimensions are spatial and the fourth dimension is due to the mass. The space of mass-points encompasses together in one space both the vectors and the points from three dimensions: the vectors v are isomorphic (equivalent to) the mass-points $(v, 0)$ with zero mass, the points P are isomorphic to the mass-points $(P, 1)$ with unit mass. Typically, we shall write

$$(mP, m) \equiv (P, 1)$$

to denote that (mP, m) and $(P, 1)$ have the same location.

The space of mass-points is related to projective three-space, but these two spaces are not the same. A point in projective three-space is an equivalence class of mass-points, where two mass-points are equivalent if they have the same location in affine space, but possibly different masses. Projective three-space is not a vector space, since addition is not well-defined on equivalence classes. The mass of a mass-point plays a role somewhat similar to the fourth, homogeneous coordinate of points in projective space, since dividing by the mass does not change the location of a mass-point. Thus much of the mathematical formalism of the space of mass-points, including for example projective transformations, is similar to the mathematical formalism of projective three-space and homogeneous coordinates, but the algebra of mass-points is much richer than the algebra of projective space.

The one disadvantage of the space of mass-points is that this space is 4-dimensional, whereas the visual world of computer graphics is 3-dimensional. This fourth dimension, however, is mass-like, not spatial, so it is easy to keep track of this extra dimension. One extra dimension turns out to be a small price to pay for a coherent algebra and a geometry with three spatial dimensions that can represent both points and vectors as well as affine and projective transformations.

The space of mass-points is a standard model for the computations of contemporary computer graphics (for further details, see Goldman, 2002). In this standard model, a quaternion is simply an element in the space of mass-points. We shall demonstrate the power and efficacy of this interpretation of quaternions as mass-points in Sections 6.1 and 6.2, where we show how to apply sandwiching to compute rotation and mirror image on points as well as on vectors. But by far the most novel use of this interpretation of quaternions as mass-points appears in Section 6.3 where we show how to apply sandwiching with quaternions to compute perspective projections on points in three dimensions.

3.2 ARROWS IN FOUR DIMENSIONS

Just as we model vectors as arrows in three dimensions, we can also model mass-points as arrows in four dimensions (see Figure 3.3). Notice that the arrows $(v, 0)$ correspond to vectors in three dimensions, the arrows $(P, 1)$ correspond to points in three dimensions, and the arrows (mP, m) represent the point P together with a mass $m \neq 0$. The usual triangle rules for addition, subtraction, and scalar multiplication of arrows (Figure 1.1) are equivalent to Equations (3.1)–(3.6) for addition, subtraction, and scalar multiplication for mass-points.

We shall use the symbol O to denote the origin for the points in three dimensions and the symbol ϑ to denote the zero vector in both three and four dimensions. Notice that the vectors $v \equiv (v, 0)$ in three dimensions are all orthogonal to the arrow representing the mass-point O in four dimensions.

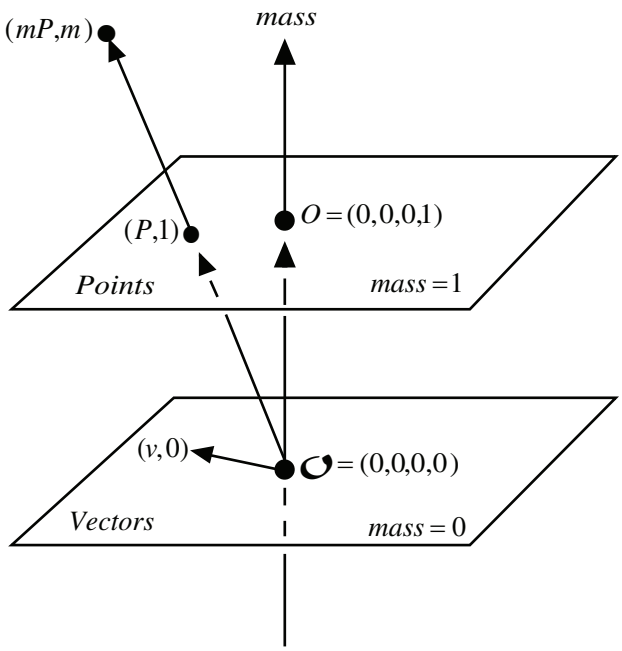
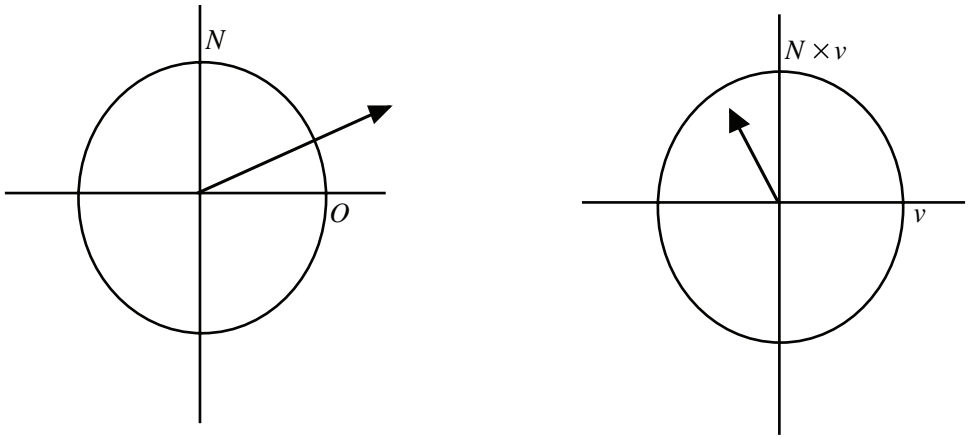


FIGURE 3.3: The space of mass-points represented as arrows in four dimensions. The symbol O denotes the origin for the points in three dimensions and the symbol ϑ denotes the zero vector in both three and four dimensions.

3.3 MUTUALLY ORTHOGONAL PLANES IN FOUR DIMENSIONS

To represent the 4-dimensional space of mass-points by arrows in four dimensions, we necessarily have to crop one or two dimensions to draw Figure 3.3 on a flat page. Thus even though we indicate each mass-point in this figure by four coordinates, at best we can actually see only two or three dimensions. To see all four dimensions at the same time, we now introduce yet a third way to visualize quaternions by looking at two mutually orthogonal planes in four dimensions.

Let $N = (N, 0)$ be a unit vector in three dimensions and let $v = (v, 0)$ be a unit vector in three dimensions orthogonal to N. Since in the 4-dimensional space of mass-points every vector $v = (v, 0)$ is orthogonal to the origin O of the points in affine space, the two planes in four dimensions spanned by O, N and v, $N \times v$ are mutually orthogonal: every mass-point in the plane spanned by O, N is orthogonal to every vector in the plane spanned by v, $N \times v$. Hence every mass-point is the sum of its components in these two mutually orthogonal planes. Therefore we can visualize simultaneously all four dimensions of any mass-point by modeling its projections into these two mutually orthogonal planes (see Figure 3.4).



(a) the plane of O, N *(b) the plane perpendicular to O, N*

FIGURE 3.4: A mass-point represented by its projections (arrows) into two mutually orthogonal planes in four dimensions.

Although Figure 3.4 depicts two planes in four dimensions, the geometric interpretations of these two planes in terms of mass-points are markedly different: the plane perpendicular to O, N represents a plane of vectors in three dimensions, but in three dimensions the plane of O, N represents a line of points.

Consider first the plane perpendicular to O, N. Since the vectors v and $N \times v$ are orthogonal in three dimensions, the plane in four dimensions of the vectors

$$w = av + bN \times v$$

spanned by v, $N \times v$ represents the plane of 3-dimensional vectors perpendicular to N—see Figure 3.5.

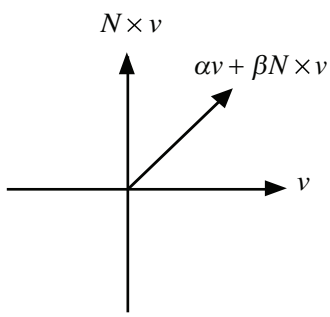
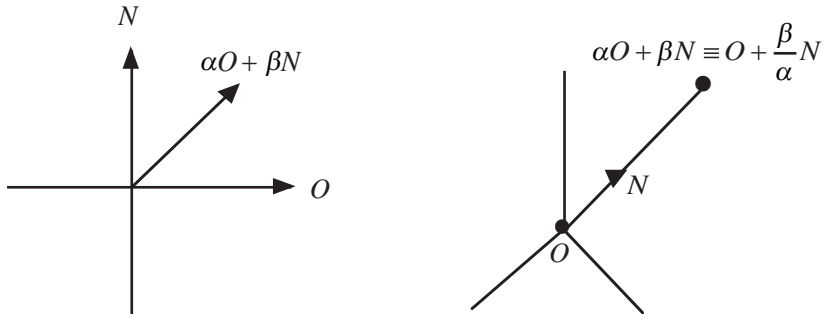


FIGURE 3.5: The plane of vectors perpendicular to O, N in four dimensions is the plane of vectors in three dimensions perpendicular to N. This plane is spanned by the vectors v and $N \times v$, where v is any nonzero vector in three dimensions perpendicular to N.



(a) a plane of vectors in 4-dimensions (b) a line of points in 3-dimensions.

FIGURE 3.6: (a) The plane of vectors (mass-points) in four dimensions spanned by O, N is equivalent to (b) the line of affine points in three dimensions passing through the point O in the direction of the vector N. This line is actually 2-dimensional, but the second dimension is mass-like, not spatial, so this dimension is not visible in (b).

On the other hand, consider the plane of O, N. In three dimensions O represents a point, not a vector. Thus the plane of 4-dimensional vectors

$$P = cO + sN$$

spanned by O, N actually represents a line in three dimensions: the line through the affine point O in the direction of the unit vector N. In fact,

$$P \equiv O + \tfrac{s}{c}N$$

—that is, P is a mass-point with mass $m = c$ on the line $P(t) = O + tN$. (If $c = 0$, then P is a vector sN parallel to the line $P(t)$.) The plane spanned by O, N does have two dimensions, but only one dimension is spatial; the other dimension, the coefficient of O, represents mass not length (see Figure 3.6).

In addition to this geometric distinction, there is also an algebraic distinction between these two mutually orthogonal planes in four dimensions. We shall see in Chapter 5 that the plane of O, N is isomorphic algebraically to the complex numbers C, whereas the plane perpendicular to O, N is isomorphic not to C but to Cv, where v is a unit vector perpendicular to N. We are going to take advantage of these asymmetries in the algebraic and geometric properties of these two mutually orthogonal planes in four dimensions in Chapter 6 when we study affine and projective transformations on points and vectors in three dimensions.

Other techniques for visualizing quaternions are presented by Hanson [2006], but for our purposes the three methods presented here will suffice.

Armed with these three geometric models for quaternions, we are now ready to take on the algebra and geometry of quaternion multiplication. In Chapter 4 we shall develop the algebra of quaternion multiplication. After we understand this algebra, in Chapter 5 we will provide a geometric way to visualize quaternion multiplication.

• • • •

CHAPTER 4

The Algebra of Quaternion Multiplication

Quaternions are just another name for the entities in the 4-dimensional space of mass-points, just as *complex numbers* are just another name for the entities in the 2-dimensional space of vectors in the plane. In fact, quaternions are an extension of complex numbers to four dimensions, since we can multiply two quaternions in a manner similar to the way that we can multiply two complex numbers. In this chapter, we are going to derive the formula for quaternion multiplication.

To multiply two complex numbers, we need to choose a preferred direction: the direction of the identity vector for multiplication in the plane. Similarly, to multiply two mass-points, we must choose a preferred mass-point: the identity for multiplication in four dimensions. We shall denote this special mass-point by O, and we will think of this point as the origin, not in the 4-dimensional vector space of mass-points, but rather in the 3-dimensional affine space of affine points. Thus O is a special mass-point with mass $m = 1$ (see Figure 3.3). The choice of the point O is arbitrary, just as the choice of the origin is arbitrary, but to perform multiplication with mass-points, we must fix O once and for all. In four dimensions, the 3-dimensional vectors lie in a subspace orthogonal to O. Hence if we were to introduce rectangular coordinates into the 4-dimensional space of mass-points, then we would have $O = (0, 0, 0, 1)$, and the 3-dimensional vectors v would be written as $v = (v_1, v_2, v_3, 0)$. Thus our choice of the point O as the identity for quaternion multiplication will not bias our definition of how multiplication should behave on vectors in three dimensions.

To fix our notation, from here on we shall typically use lower case letters p, q, r from the middle of the alphabet to represent arbitrary mass-points—that is, arbitrary quaternions—and lower case letters u, v, w from the end of the alphabet to represent vectors in three dimensions —that is, quaternions with zero mass. Uppercase letters O, P, Q from the middle of the alphabet will be used to denote points in 3-dimensional affine space—that is, quaternions with mass equal to 1.

Every mass-point $(mP, m) = mO + v$, where $v = m(P - O)$ is a vector in three dimensions. The term mO designates the mass m; the vector v specifies the location $O + v/m$. Interpreted in terms of coordinates, the equation $(mP, m) = mO + v$ means that every quaternion q can be written using four coordinates—that is, as $q = (q_1, q_2, q_3, q_4)$ or equivalently

$$q = q_4 O + q_1 i + q_2 j + q_3 k.$$

In many texts the letter O is dropped, and quaternions are written as

$$q = q_4 + q_1 i + q_2 j + q_3 k.$$

In this notation, a quaternion q seems to be a weird kind of hermaphrodite, a scalar q_4 added to a vector $q_1 i + q_2 j + q_3 k$. In fact, there are no mysterious hermaphrodites; the quaternion q in this equation is simply a mass-point with mass q_4.

Now we expect quaternion multiplication to have the following properties:

- O is the identity

  $Op = pO = p$
- Multiplication is associative and distributes through addition

  $p(qr) = (pq)r$

  $p(q + r) = pq + pr$

  $(q + r)p = qp + rp$

These assumptions immediately imply that

$$(aO + v)(bO + w) = (ab)O + aw + bv + vw. \tag{4.1}$$

Equation (4.1) tells us that to complete the definition of multiplication on the 4-dimensional space of mass-points, we need only define multiplication vw on the subspace of vectors in three dimensions. To derive the formula for the product of two vectors, we need to make one additional assumption.

For two complex numbers z_1, z_2, we know that

$$|z_1 z_2| = |z_1|\,|z_2|$$

—that is, the length of the product is equal to the product of the lengths. Therefore multiplication by a complex number w scales the length of all other complex numbers—that is, scales the length of all vectors in the plane—by the same amount. Thus under complex multiplication, triangles are mapped to similar triangles, so angles are preserved by complex multiplication. Hence complex multiplication is a linear transformation that preserves angles, so complex multiplication is a conformal map. Suppose we assume that quaternion multiplication has the analogous property:

$$|q_1 q_2| = |q_1|\,|q_2|.$$

Then quaternion multiplication would be a conformal map on vectors in four dimensions—that is, quaternion multiplication would also preserve angles. With this assumption, we can now derive the following properties of the product vw of two vectors in three dimensions.

Proposition 4.1: $vw \perp v, w$ *(4.2)*

Proof: Since multiplication preserves angles:

$$w \perp O \Rightarrow vw \perp vO = v$$

$$v \perp O \Rightarrow vw \perp Ow = w. \qquad \blacklozenge$$

Proposition 4.2: $v^2 = -(v \cdot v)O$ *(4.3)*

Proof: Let $q = \cos(\theta)O + \sin(\theta)v$, with $\theta \neq k(\pi/2)$. Since multiplication preserves angles,

$$v \perp O \Rightarrow qv \perp qO = q,$$

so

$$qv = (\cos(\theta)O + sin(\theta)v)v = \cos(\theta)v + \sin(\theta)v^2 \perp \cos(\theta)O + \sin(\theta)v = q.$$

But by Proposition 4.1, $v^2 \perp v$. Therefore there exist scalars α, β such that

$$v^2 = \alpha O + \beta w \qquad w \perp O, v\,.$$

Hence

$$0 = qv \cdot q = (\cos(\theta)v + \sin(\theta)(\alpha O + \beta w)) \cdot (\cos(\theta)O + \sin(\theta)v) = (v \cdot v + \alpha)\sin(\theta)\cos(\theta),$$

so

$$\alpha = -v \cdot v\,.$$

Moreover $\beta = 0$ because

$$\alpha^2 = (v \cdot v)^2 = |v|^4 = |v^2|^2 = \alpha^2 + \beta^2. \qquad \blacklozenge$$

Corollary 4.1: *angle between O and* vw = *angle between* $-v$ *and* w.

Proof: Since multiplication preserves angles:

angle between O and vw = angle between $-v$ and $(-v)(vw) = -v^2 w = (v \cdot v)w \,||\, w$. $\blacklozenge$

Corollary 4.2: $v \perp w \Leftrightarrow vw \perp O$. *(4.4)*

Proof: By Corollary 4.1,

angle between O and vw = angle between $-v$ and w. $\blacklozenge$

Corollary 4.3: $(vw) \cdot O = -v \cdot w$. *(4.5)*

Proof: Let θ is the angle between v and w. Then by Corollary 4.1:

angle between O and vw = angle between $-v$ and $w = \pi - \theta$.

Therefore

$$(vw)\cdot O = |v\ w|\ |O|\cos(\pi-\theta) = -|v|\ |w|\cos(\theta) = -v\cdot w. \qquad \blacklozenge$$

To summarize, we have derived the following consequences of conformality:

- $vw \perp v, w$ *(4.2)*
- $v^2 = -(v\cdot v)O$ *(4.3)*
- $v \perp w \Leftrightarrow vw \perp O$ *(4.4)*
- $(vw)\cdot O = -v\cdot w$ *(4.5)*

Using these results, we can show that there can be no multiplication in three dimensions. For let i, j be two orthogonal unit vectors both perpendicular to the identity O. Then by Equations (4.2) and (4.4),

$$ij \perp O, i, j$$

But $|ij| = |i|\ |j| = 1$. Hence, O, i, j, ij would be four mutually orthogonal nonzero vectors, which is not possible in three dimensions. Similar arguments can be used to show that there is no multiplication in any dimension greater than 4 [Hurwitz, 1898] (see also Exercises 4.9–4.12). (A nonassociative rule for multiplication exists in eight dimensions; these numbers are called *octonions* (see Conway and Smith, 2003). Of course in four dimensions we could accommodate the product ij by setting $ij = k$. In fact, using Equations (4.2)–(4.5), we can now derive an explicit formula for the product of two vectors.

Proposition 4.3: $vw = (-v\cdot w)O + v\times w$ *(4.6)*

Proof: By Equation (4.2), $vw \perp v, w$. Therefore since O and $v\times w$ span the plane of vectors in four dimensions perpendicular to v, w, there must be constants a, b, such that

$$vw = aO + bv\times w. \tag{4.7}$$

Moreover, by Equation (4.5),

$$(vw)\cdot O = -v\cdot w,$$

so dotting both sides of Equation (4.7) with O yields

$$a = -v\cdot w.$$

Now since $v\times w \perp O$, it follows by the Pythagorean theorem that

$$|vw|^2 = |-(v\cdot w)O + bv\times w|^2 = (v\cdot w)^2 + b^2\,|v\times w|^2,$$

so

$$|v|^2\,|w|^2 = |v|^2\,|w|^2\,(\cos^2(\theta) + b^2\sin^2(\theta)),$$

where θ is the angle between v and w. Canceling $|v|^2\,|w|^2$ on both sides, we arrive at

$$1 = \cos^2(\theta) + b^2\sin^2(\theta).$$

We conclude that $b = \pm 1$. This sign ambiguity reflects the fact that we could choose the cross product to satisfy either the left or the right-hand rule. We choose the right-hand rule and set $b = +1$. Thus

$$vw = (-v \cdot w)O + v \times w. \qquad \blacklozenge$$

Combining Equations (4.1) and (4.6), we arrive finally at the formula:

$$(aO + v)(bO + w) = (ab - v \cdot w)O + aw + bv + v \times w. \tag{4.8}$$

Equation (4.8) is the definition of the product of two arbitrary quaternions. Notice how the vector products—dot product, cross product, and scalar product—all appear in the formula for quaternion multiplication. (In fact, dot product and cross product can be defined in terms of the quaternion product; see Exercise 4.2.) Except for the choice of the special point O, quaternion multiplication is coordinate-free, since dot product, cross product, and scalar product are all coordinate-free. Using Equation (4.8), it is straightforward, although somewhat tedious, to verify that quaternion multiplication is associative and distributes through addition (see Exercise 4.1). Notice, however, that quaternion multiplication is neither commutative nor anticommutative, since

$$v \cdot w = w \cdot v$$

but

$$v \times w = -w \times v.$$

The point O serves as the identity for quaternion multiplication, so

$$O^2 = O,$$

and more generally for any quaternion p, we have

$$Op = p = pO.$$

Moreover, by Equation (4.6),

$$|u| = 1 \Rightarrow u^2 = -O \tag{4.9}$$

$$v \perp w \Rightarrow vw = v \times w. \tag{4.10}$$

Therefore the basis vectors i, j, k multiply according to the following rules:

$$i^2 = j^2 = k^2 = -O \text{ and } ij = k = -ji, jk = i = -kj, ki = j = -ik. \tag{4.11}$$

Thus if $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$, then by Equation (4.11) (see Exercise 4.3)

$$\begin{aligned} pq &= (p_4O + p_1i + p_2j + p_3k)(q_4O + q_1i + q_2j + q_3k) \\ &= (p_4q_4 - p_1q_1 - p_2q_2 - p_3q_3)O + (p_4q_1 + p_1q_4 + p_2q_3 - p_3q_2)i \\ &\quad + (p_4q_2 + p_2q_4 + p_3q_1 - p_1q_3)j + (p_4q_3 + p_3q_4 + p_1q_2 - p_2q_1)k \end{aligned} \tag{4.12}$$

A more concise formula for quaternion multiplication can be given by representing each quaternion by two complex numbers instead of by four real numbers. To express a quaternion q by a pair of complex numbers, observe that the quaternions $\alpha = aO + bi$ are isomorphic to the complex numbers, since $O^2 = O$ and $i^2 = -O$, so we shall identify the quaternions $\alpha = aO + bi$ with the complex numbers. Now by Equation (4.11)

$$q = q_4O + q_1i + q_2j + q_3k = (q_4O + q_1i) + (q_2O + q_3i)j,$$

so every quaternion q can be expressed uniquely as

$$q = \alpha_q + \beta_q j,$$

where α_q , β_q are complex numbers. Moreover, notice that for any complex number $\alpha = aO + bi$

$$j\alpha = \alpha^* j. \tag{4.13}$$

This identity allows us to express quaternion multiplication concisely in terms of complex multiplication:

$$pq = (\alpha_p + \beta_p j)(\alpha_q + \beta_p j) = (\alpha_p\alpha_q - \beta_p\beta_q^*) + (\alpha_p\beta_q + \beta_p\alpha_q^*)j. \tag{4.14}$$

We shall have a good deal more to say about the relationship between complex multiplication and quaternion multiplication in Chapter 5—see especially Remark 5.4.

Every nonzero quaternion has a multiplicative inverse. Let $q = m_qO + v_q$. The conjugate of q, denoted by q^*, is defined by

$$q^* = m_qO - v_q.$$

By Equation (4.8)

$$qq^* = \left(m_q^2 + v_q \cdot v_q\right) O.$$

But recall that in quaternion space $v_q \perp O$, so by the Pythagorean theorem

$$m_q^2 + v_q \cdot v_q = m_q^2 + |v_q|^2 = |q|^2.$$

Hence

$$qq^* = |q|^2 O. \tag{4.15}$$

Therefore

$$q^{-1} = \frac{q^*}{|q|^2}.$$

In terms of coordinates, if $q = (q_1, q_2, q_3, q_4)$, then

$$q = q_4 O + q_1 i + q_2 j + q_3 k$$

$$q^* = q_4 O - q_1 i - q_2 j - q_3 k$$

$$|q|^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

Every quaternion can be written as

$$q = m_q O + v_q = m_q O + n_q u_q,$$

where n_q is a scalar, u_q is a unit vector, and $n_q u_q = v_q$. If q is a unit quaternion, then since $u_q \perp O$ it follows by the Pythagorean theorem that

$$m_q^2 + n_q^2 = |q|^2 = 1.$$

Therefore there is an angle θ such that

$$m_q = cos(\theta)$$

$$n_q = sin(\theta).$$

Hence every unit quaternion can be expressed as

$$q(u, \theta) = \cos(\theta) O + \sin(\theta) u, \tag{4.16}$$

where $\cos(\theta) = m_q$ is the mass, u is a unit vector, and $\sin(\theta) u = v_q$. Thus for unit quaternions $q(u, \theta)$, the conjugate is given by

$$q^*(u, \theta) = \cos(\theta) O - \sin(\theta) u = q(u, -\theta) \tag{4.17}$$

(see too Exercise 4.6).

Using the conjugate, we can now verify that the length of the product of two quaternions is indeed equal to the product of their lengths. First we need the following result.

Proposition 4.4: $(pq)^* = q^*p^*$ *(4.18)*

Proof: Let $p = m_pO + v_p$ and $q = m_qO + v_q$. Then by Equation (4.8):

$$pq = (m_pO + v_p)(m_qO + v_q) = (m_pm_q - v_p \cdot v_q)O + (m_qv_p + m_pv_q + v_p \times v_q)$$
$$q^*p^* = (m_qO - v_q)(m_pO - v_p) = (m_qm_p - v_q \cdot v_p)O + (-m_qv_p - m_pv_q + v_q \times v_p).$$

Comparing the right-hand sides of these two equations and recalling that the cross product is anti-commutative, we conclude that

$$(pq)^* = q^*p^*. \qquad \blacklozenge$$

Corollary 4.4: $|pq| = |p|\,|q|$ *(4.19)*

Proof: By Proposition 4.4 and Equation (4.15)

$$|pq|^2\,O = (pq)(pq)^* = (pq)(q^*p^*) = p(qq^*)p^* = |p|^2\,|q|^2O.$$

Hence since length is nonnegative

$$|pq| = |p|\,|q|.$$

(For an alternative proof, see Exercise 4.5.) ♦

By Corollary 4.4, the length of the product of two quaternions is indeed equal to the product of their lengths. Since quaternion multiplication distributes through addition, multiplication by a unit quaternion is a linear transformation that preserves lengths (a linear isometry), so we can expect multiplication by unit quaternions to represent rotations in four dimensions. In the next chapter, we are going to use this observation to visualize the geometric effects of quaternion multiplication.

Exercises:

4.1. Using Equation (4.8) and the properties of dot product and cross product, verify that quaternion multiplication is associative and distributes through addition.

4.2. Show that

a. $(N \cdot v)O = \dfrac{-(vN + Nv)}{2}$

b $N \times v = \dfrac{Nv - vN}{2}$.

4.3. Let $p = p_4O + p_1i + p_2j + p_3k$ and $q = q_4O + q_1i + q_2j + q_3k$. Verify that

$$pq = (p_4q_4 - p_1q_1 - p_2q_2 - p_3q_3)O + (p_4q_1 + p_1q_4)i + (p_4q_2 + p_2q_4)j + (p_4q_3 + p_3q_4)k$$
$$+ (p_2q_3 - p_3q_2)i + (p_3q_1 - p_1q_3)j + (p_1q_2 - p_2q_1)k.$$

4.4. Let v,w be vectors in three dimensions (i.e., quaternions with zero mass). Show that

a. $v^2 + w^2 = (v + w)^2$ if v is perpendicular to w.

b. $v^2 - w^2 \neq (v + w)(v - w)$ if v is not parallel to w.

c. Conclude from part b that it is possible for $v^2 = w^2$, but $v \neq \pm w$.

d. Give an example where $v^2 = w^2$, but $v \neq \pm w$.

4.5. Let a,b,c,d,w,x,y,z be real numbers, and let $q_1 = aO + bi + cj + dk$, $q_2 = wO + xi + yj + zk$ be quaternions.

a. Verify the following generalization of Brahmagupta's identity:

$$(a^2 + b^2 + c^2 + d^2)(w^2 + x^2 + y^2 + z^2) = (aw - bx - cy - dz)^2 + (ax + bw + cz - dy)^2$$
$$+ (ay - bz + cw + dx)^2 + (az + by - cx + dw)^2.$$

b. Using the result in part a, prove that

$$|q_1q_2| = |q_1|\,|q_2|.$$

4.6. Show that

$$q(N,\phi)^{-1} = q(N,\phi)^* = q(N,-\phi) = q(-N,\phi)$$

4.7. Let $q = q(N,\theta) = (q_1, q_2, q_3, q_4)$ be a unit quaternion. Show that

$$\sqrt{q} = q(N, \theta/2) = \frac{1+q}{\sqrt{2(1+q_4)}}.$$

4.8. Let p, q be unit quaternions. Show that

$$p \cdot q = (pq^*)_4$$

4.9. Prove that there no viable multiplication in five dimensions.

Hint:

a. Let O,i,j be orthogonal unit vectors.

b. Define $k = ij$.

c. Let $l \perp O, i, j, k$.

d. Consider il.

4.10. Prove that there no viable multiplication in six or seven dimensions.

(Hint: Proceed as in Exercise 4.9 and consider il, jl, kl.)

4.11. Generalize the argument in Exercise 4.10 to show that there is no viable multiplication in R^d for $d \neq 2^p$.

4.12. In this exercise we shall show that there is no associative rule for multiplication in any dimension $d > 4$.

a. Let O denote the identity for multiplication, and let v, w be orthogonal vectors. Using Equation (4.3) and the associative property of multiplication, show that $v, w \perp O \Rightarrow v\,w = -w\,v$.

b. Let O, i, j be orthogonal unit vectors, and define $k = i\,j$. Using part a and Equation (4.2), show that $span(O, i, j, k)$ is isomorphic to the quaternions.

c. Let $l \perp O, i, j, k$. Show that $q\,l = l\,q^*$ for any quaternion $q \in span(O, i, j, k)$.

d. Use part c to express $p\,l\,q^*$ in two ways and conclude that $p\,q = q\,p$ for any two quaternions p, q.

e. Conclude from part d that there is no associative rule for multiplication in any dimension $d > 4$.

CHAPTER 5

The Geometry of Quaternion Multiplication

We want to see what happens when we multiply an arbitrary quaternion p by a unit quaternion

$$q(N, \theta) = cos(\theta)O + sin(\theta)N,$$

where N is a unit vector and O is the identity for quaternion multiplication. There is one key algebraic insight that will allow us to visualize the geometric effects of quaternion multiplication: *For every unit vector N, the plane of O, N is isomorphic to the complex plane*.

Indeed, by Equation (4.9), for any unit vector N, we have

$$N^2 = -O.$$

Since

$$O^2 = O,\ ON = NO = N, \text{ and } N \perp O,$$

multiplication in the plane determined by the two quaternions O, N behaves exactly like multiplication in the complex plane. Therefore, just as multiplication by the complex number

$$e^{i\theta} = \cos(\theta) + \sin(\theta)i$$

rotates vectors in the complex plane by the angle θ, multiplication by the unit quaternion

$$q(N, \theta) = \cos(\theta)\ O + \sin(\theta)N$$

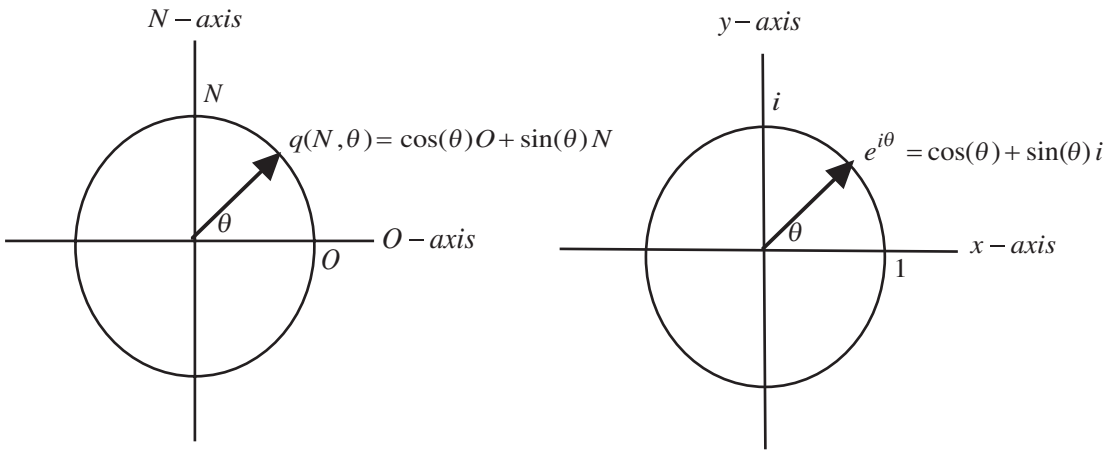
rotates quaternions in the plane of O, N by the angle θ (see Figure 5.1).

Therefore, we have the following results.

Lemma 5.1: *Let N be a unit vector. Then*

i. $q(N, \theta)\, q(N, \phi) = q(N, \theta + \phi)$

ii. $q(N, \theta)\, q(N, \phi) = q(N, \phi)\, q(N, \theta)$



(a) the quaternion plane of O, N *(b) the complex plane*

FIGURE 5.1: For any unit vector N, (a) the plane determined by O, N is isomorphic algebraically and homeomorphic geometrically to (b) the complex plane: $O \leftrightarrow 1$ and $N \leftrightarrow i$, since $O^2 = O$, $N^2 = -O$, $ON = NO = N$, and $N \perp O$. Hence, $q(N, \theta) = \cos(\theta)\, O + \sin(\theta) N \leftrightarrow e^{i\theta} = \cos(\theta) + \sin(\theta) i$.

Proof: To prove *i*, we simply apply quaternion multiplication and the trigonometric identities for the sine and the cosine of the sum of two angles:

i. $$\begin{aligned} q(N, \theta)\, q(N, \phi) &= (cos(\theta)O + sin(\theta)N)(cos(\phi)O + sin(\phi)N) \\ &= (cos(\theta)cos(\phi) - sin(\theta)sin(\phi))O + (sin(\theta)cos(\phi) + cos(\theta)sin(\phi))N \\ &= cos(\theta + \phi)O + sin(\theta + \phi)N \\ &= q(N, \theta + \phi). \end{aligned}$$

ii. Follow immediately from *i*. ♦

Since multiplying arbitrary quaternions by a fixed unit quaternion is a linear isometry in four dimensions, unit quaternions $q(N, \theta)$ represent rotations in four dimensions. By Lemma 5.1, we understand how $q(N, \theta)$ rotates quaternions in the plane determined by O, N because in this plane multiplication by $q(N, \theta)$ behaves just like multiplication by $e^{i\theta}$ in the complex plane. But to fully visualize how multiplication by $q(N, \theta)$ affects arbitrary quaternions, we still need to see how $q(N, \theta)$ affects quaternions in the complementary plane of quaternions perpendicular to O, N.

Now the key observations are simply that

- $p \parallel O, N \Rightarrow q(N, \theta)p,\ p\, q(N, \theta) \parallel O, N$
- $v \perp O, N \Rightarrow q(N, \theta)v,\ v\, q(N, \theta) \perp O, N.$

That is, multiplying by $q(N, \theta)$ maps each of these planes—the plane parallel to O, N and the plane perpendicular to O, N—into themselves. The first observation follows immediately from Lemma 5.1. The second observation holds because multiplication by unit quaternions on the left or the right is an isometry and hence preserves angles; therefore

$$v \perp O, N \Rightarrow q(N, \theta)\, v \perp q(N, \theta)O,\ q(N, \theta)N \Rightarrow q(N, \theta)v \perp q(N, \theta),\ q(N, \theta + \pi/2)$$

and

$$v \perp O, N \Rightarrow v\, q(N, \theta) \perp O\, q(N, \theta),\ N\, q(N, \theta) \Rightarrow v\, q(N, \theta) \perp q(N, \theta),\ q(N, \theta + \pi/2).$$

Thus quaternion multiplication represents a double rotation, a rotation in two orthogonal planes.

Hence since every quaternion can be decomposed into a quaternion in the plane of O, N and a vector in the plane perpendicular to O, N, the essential insight is that we can study the rotations in four dimensions induced by multiplication with the unit quaternions $q(N, \theta)$ by studying the effect of multiplication with $q(N, \theta)$ in the plane of O, N, where by Lemma 5.1 we know that multiplication is commutative, and by studying the effect of multiplication with $q(N, \theta)$ in the complementary plane of vectors perpendicular to O, N, where we know that multiplication cannot be commutative, since otherwise multiplication with $q(N, \theta)$ would commute with every quaternion. The different effects of the 4-dimensional rotation represented by multiplication with $q(N, \theta)$ in these two complementary planes will allow us to visualize quaternion multiplication in four dimensions.

In facilitate our presentation, we introduce the following notation:

$L_q(p) = qp$	left multiplication by q
$R_q(p) = pq$	right multiplication by q
$T_q(p) = qpq = L_q(R_q(p))$	sandwiching p between two copies of q
$S_q(p) = qpq^* = L_q(R_{q^*}(p))$	sandwiching p between q and q^*

The functions $L_q(p)$ and $R_q(p)$ are linear transformations on the vector space of quaternions because quaternion multiplication distributes through quaternion addition. The sandwiching operators $T_q(p)$ and $S_q(p)$ are also linear transformations because they are composites of linear transformations. These two sandwiching operators measure the noncommutativity of quaternion multiplication. We shall see in Section 6.1 that the sandwiching operator $S_q(p)$ plays a central role in the use of quaternions for computing rotations in three dimensions. In Sections 6.2 and 6.3, we shall see how to use the sandwiching operator $T_q(p)$ to compute reflections and perspective projections.

We are now ready to study the geometric effects of multiplication by the unit quaternions

$$q(N, \theta) = \cos(\theta)O + \sin(\theta)N$$

on quaternions in the plane of O, N—which we have already seen is isomorphic to the complex plane—and on quaternions in the complementary plane of vectors perpendicular to O, N.

Proposition 5.2: *Rotation in the plane of O, N*

Let

- $N =$ *a unit vector*
- $p =$ *a quaternion in the plane of O, N*

Then

i. $L_{q(N,\,\theta)}(p) = q(N, \theta)p$ *rotates p by the angle θ in the plane of O, N*

ii. $R_{q(N,\,\theta)}(p) = p\,q(N, \theta)$ *rotates p by the angle θ in the plane of O, N*

iii. $R_{q^*(N,\,\theta)}p = p\,q^*(N, \theta)$ *rotates p by the angle $-\theta$ in the plane of O, N*

Proof: i, ii. Follow immediately from Lemma 5.1.

iii. Follows immediately from ii and Equation (4.17). ♦

Proposition 5.3: *Rotation in the plane perpendicular to O, N*

Let

- $N =$ *a unit vector*
- $v =$ *a vector in the plane $\perp O, N$*

Then

i. $L_{q(N,\,\theta)}(v) = q(N, \theta)v$ *rotates v by the angle θ in the plane $\perp O, N$*

ii. $R_{q(N,\,\theta)}(v) = v\,q(N, \theta)$ *rotates v by the angle $-\theta$ in the plane $\perp O, N$*

iii. $R_{q^*(N,\,\theta)}(v) = v\,q^*(N, \theta)$ *rotates v by the angle θ in the plane $\perp O, N$*

Proof: Since $v \perp N$, it follows by Equation (4.10) that $Nv = N \times v$ and $vN = v \times N$. Therefore,

i. $q(N, \theta)v = (\cos(\theta)O + \sin(\theta)N)v = \cos(\theta)v + \sin(\theta)N \times v$

ii. $v\,q(N, \theta) = v(\cos(\theta)O + \sin(\theta)N) = \cos(\theta)v + \sin(\theta)v \times N = \cos(\theta)v - \sin(\theta)N \times v$

iii. $v\,q^*(N, \theta) = v(\cos(\theta)O - \sin(\theta)N) = \cos(\theta)v - \sin(\theta)v \times N = \cos(\theta)v + \sin(\theta)N \times v$

Equations i and iii represent rotation by the angle θ and Equation ii represents rotation by the angle $-\theta$ in the plane $\perp O, N$ (see Figure 5.2). ♦

Remark 5.4: Let N be a unit vector, and let v be a unit vector perpendicular to N. Then the quaternions $O, N, v, N \times v$ represent a basis for the quaternions isomorphic algebraically to the canonical basis O, i, j, k—that is,

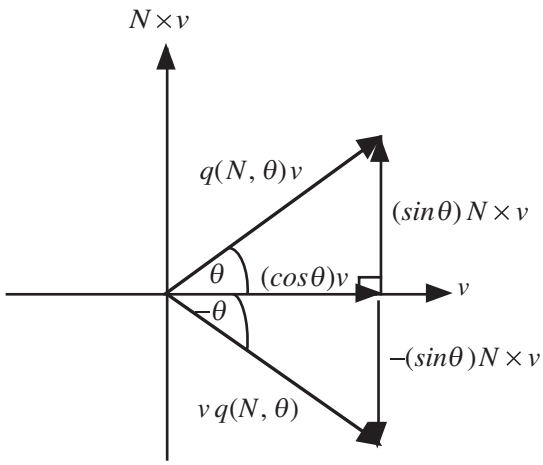


FIGURE 5.2: Rotation by the angle $\pm\theta$ in the plane $\perp$ O, N.

$$O \leftrightarrow O,\ N \leftrightarrow i,\ v \leftrightarrow j,\ N \times v \leftrightarrow k.$$

Indeed, since N and v are orthogonal unit vectors, it follows by Equation (4.9) that

$$N^2 = v^2 = (N \times v)^2 = -O.$$

Moreover, by Equation (4.10), $Nv = N \times v = -v \times N = -vN$, so

$$Nv = N \times v, \qquad v(N \times v) = v \times (N \times v) = N, \qquad (N \times v)N = (N \times v) \times N = v.$$

Thus the algebra of the quaternions O, N, v, $N \times v$ is identical to the algebra of the quaternions O, i, j, k. We have already seen in Lemma 5.1 that the quaternions $\alpha O + \beta N$ in the plane of O, N are isomorphic algebraically to the complex numbers $\alpha + \beta i$. Similarly, the quaternions in the plane orthogonal to O, N can be written as

$$\gamma v + \delta(N \times v) = (\gamma O + \delta N)v$$

—that is, as a complex number $\gamma O + \delta N$ times the vector v. Hence every quaternion p can be expressed uniquely as

$$p = \alpha_p + \beta_p v,$$

where α_p, β_p are complex numbers in the plane of O, N. Thus, as we observed at the start of Chapter 3,

$$R^4 \cong R^2 \oplus R^2 \cong C \oplus Cv. \tag{5.1}$$

This isomorphism together with the identity

$$v\,q = q^* v, \tag{5.2}$$

which by Proposition 5.3 holds for every complex number $q = \gamma O + \delta N$ and which is the analogue of Equation (4.13), allows us to interpret quaternion multiplication in the basis $O, N, v, N \times v$ in terms of complex multiplication. From this interpretation, we can see immediately from Equation (5.1) that multiplying a vector $\beta_p v$ in the plane orthogonal to O, N by a complex number $q(N, \theta) = \cos(\theta)O + \sin(\theta)N$ on the left will behave like counterclockwise rotation by the angle θ just like multiplication on the left by $q(N, \theta)$ in the complex plane, whereas by Equation (5.2) multiplying $\beta_p v$ by $q(N, \theta)$ on the right will behave like multiplication by $q^*(N, \theta) = q(N, -\theta)$ on the left—that is, like clockwise rotation by the angle θ. In technical terms, multiplication by $q(N, \theta)$ represents a *double isoclinic rotation* on R^4 [Mebius, 2005].

Multiplying quaternions by $q(N, \theta)$ does not generate a simple rotation in R^4—that is, a rotation in a single plane in R^4—since this multiplication rotates vectors in two mutually orthogonal planes in R^4. To get simple rotations in R^4, we need to use sandwiching.

Corollary 5.2: *Sandwiching in the plane of O, N*

Let

- *N = a unit vector*
- *p = a quaternion in the plane of O, N*

Then

i. $T_{q(N,\,\theta)}(p) = q(N, \theta)\, p\, q(N, \theta)$ *rotates p by the angle 2θ in the plane of O, N*

ii. $S_{q(N,\,\theta)}(p) = q(N, \theta)\, p\, q^*(N, \theta)$ *is the identity on p.*

Proof: These results follow immediately from Proposition 5.2 ♦

Corollary 5.3: *Sandwiching in the plane perpendicular to O, N*

Let

- *N = a unit vector*
- *v = a vector in the plane $\perp O, N$*

Then

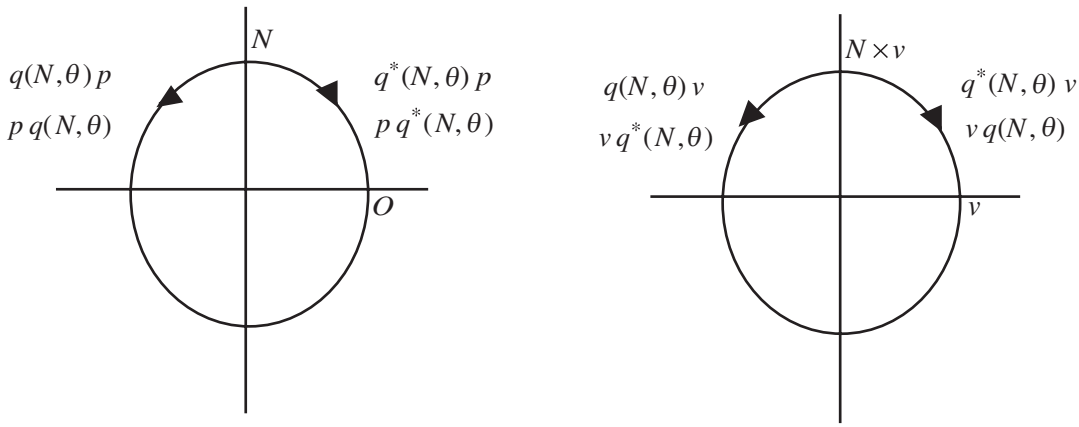
i. $T_{q(N,\,\theta)}(v) = q(N, \theta)\, v\, q(N, \theta)$ *is the identity on v*

ii. $S_{q(N,\,\theta)}(v) = q(N, \theta)\, v\, q^*(N, \theta)$ *rotates v by the angle 2θ in the plane $\perp O, N$.*

Proof: These results follow immediately from Proposition 5.3. ♦

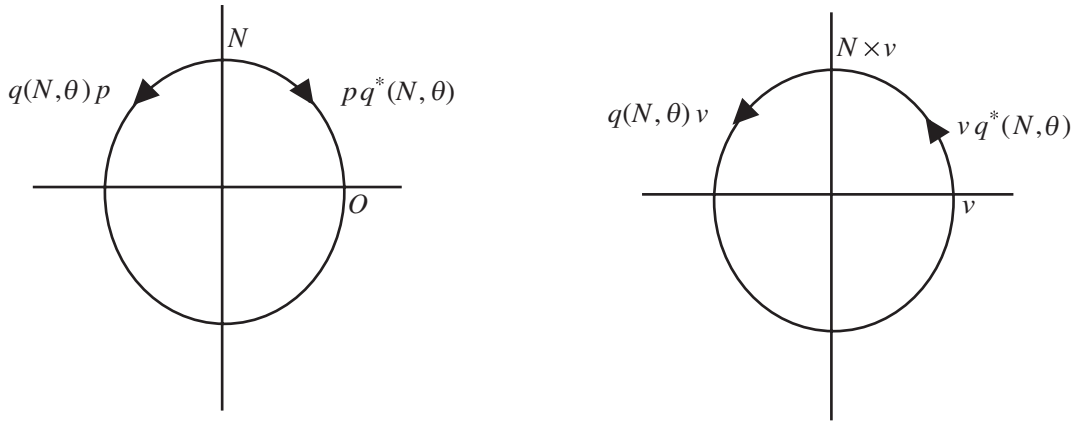
For easy reference, we summarize graphically the results of Propositions 5.2, 5.3, and Corollaries 5.2, 5.3 in Figures 5.3–5.5.

With this introduction to quaternion algebra and geometry in four dimensions, we are finally ready to turn our attention to the main applications of quaternion multiplication in computer graphics: computing affine and projective transformations on points and vectors in three dimensions.



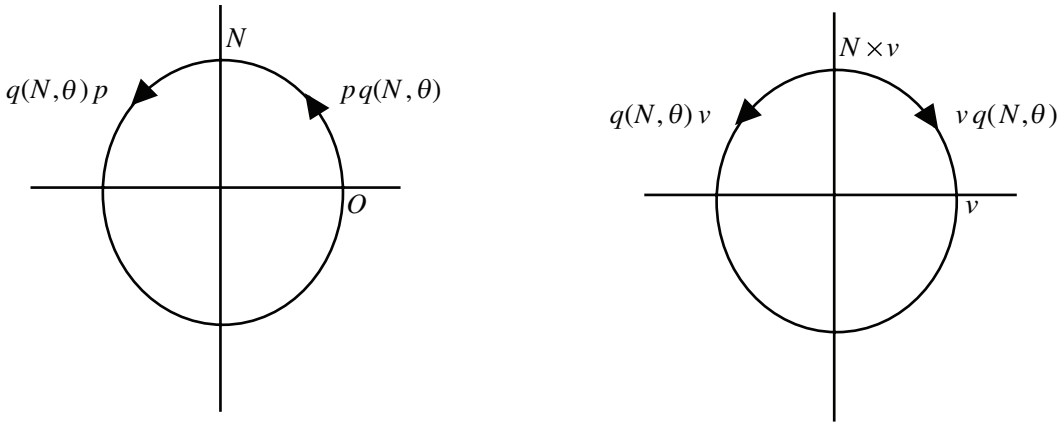
(a) the plane of O, N *(b) the plane perpendicular to O, N*

FIGURE 5.3: *Double Isoclinic Rotations in 4-Dimensions*. The effects of multiplying a quaternion on the left and right with $q(N, \theta)$ in the plane of O, N and in the complementary plane perpendicular to O, N. In (a) the plane of O, N, multiplying by $q(N, \theta)$ on the left or the right rotates quaternions p counterclockwise by the angle angle θ, whereas multiplying by the conjugate $q^*(N, \theta)$ on the left or the right rotates quaternions p in the opposite clockwise direction by the angle θ, but in (b), the plane of vectors perpendicular to the plane of O, N, multiplying by $q(N, \theta)$ on the left or by its conjugate $q^*(N, \theta)$ on the right rotates vectors v counterclockwise by the angle θ, whereas multiplying by $q(N, \theta)$ on the right or by its conjugate $q^*(N, \theta)$ on the left rotates vectors v in the opposite clockwise direction by the angle θ. Thus left and right quaternion multiplication both represent double isoclinic rotations in four dimensions.



(a) the plane of O, N *(b) the plane perpendicular to O, N*

FIGURE 5.4: *Simple Rotation in 4-Dimensions*. The effects of sandwiching a quaternion between $q(N, \theta)$ and $q^*(N, \theta)$ in the plane of O, N and in the complementary plane perpendicular to O, N. In (a) the plane of O, N, the rotation induced by multiplying with $q(N, \theta)$ on the left is canceled by the rotation induced by multiplying with its conjugate $q^*(N, \theta)$ on the right, whereas in (b) the plane of vectors perpendicular to the plane of O, N, the rotation induced by multiplying with $q(N, \theta)$ on the left is reinforced by the rotation induced by multiplying with its conjugate $q^*(N, \theta)$ on the right. Hence for an arbitrary vector u, the map $S_{q(N, \theta)}(u) = q(N, \theta)\, u\, q^*(N, \theta)$ rotates the component of u perpendicular to N by the angle 2θ and leaves the component of u parallel to N unchanged. Thus the sandwiching map $S_{q(N, \theta)}(u)$ represents a simple rotation in four dimensions.



(a) the plane of O, N *(b) the plane perpendicular to O, N*

FIGURE 5.5: *Simple Rotation in 4-Dimensions*. The effect of sandwiching a quaternion between two copies of $q(N, \theta)$ in the plane of O, N and in the complementary plane perpendicular to O, N. In (a), the plane of O, N, the rotation induced by multiplying with $q(N, \theta)$ on the left is reinforced by the rotation induced by multiplying with $q(N, \theta)$ on the right, whereas in (b) the plane of vectors perpendicular to the plane of O, N, the rotation induced by multiplying with $q(N, \theta)$ on the left is canceled by the rotation induced by multiplying with $q(N, \theta)$ on the right. Hence for an arbitrary vector u, the map $T_{q(N,\,\theta)}(u) = q(N, \theta)\, u\, q(N, \theta)$ leaves the component of u perpendicular to N unchanged and rotates the component of u parallel to N by the angle 2θ. Thus the sandwiching map $T_{q(N,\,\theta)}(u)$ represents a simple rotation in four dimensions.

CHAPTER 6

Affine, Semi-Affine, and Projective Transformations in Three Dimensions

Sandwiching with quaternions can be used to model several key transformations in 3-dimensional computer graphics, including rotation, mirror image, and perspective projection. In this chapter, we are going to investigate each of these transformations using quaternions. Note that here we shall need to distinguish carefully between our model of quaternions as vectors in four dimensions and our geometric interpretation of quaternions as mass-points in three dimensions.

Recall from Chapter 3 that every quaternion can be decomposed into the sum of two orthogonal quaternions: a quaternion in the plane of O, N, where N is a unit vector in three dimensions and a vector in the complementary plane perpendicular to O, N (see Figure 3.4). Therefore Propositions 5.2 and 5.3 clarify how multiplying by the unit quaternion

$$q(N, \theta) = cos(\theta)O + sin(\theta)N$$

on the left or the right rotates arbitrary quaternions in four dimensions. Corollaries 5.2 and 5.3 explain the effect of sandwiching on arbitrary quaternions in four dimensions, but what we really want to understand is the effects of sandwiching on points and vectors in three dimensions. So far, we have the following results about sandwiching:

$S_{q(N,\,\theta)}(p) = q(N, \theta)\, p\, q^*(N, \theta)$

- Plane of O, N in four dimensions ≡ line through O parallel to N in three dimensions
  — $S_{q(N,\,\theta)}(p)$ = IDENTITY ↔ FIXED AXIS LINE IN 3-DIMENSIONS
- Plane ⊥ O, N in four dimensions ≡ plane of vectors ⊥ N in three dimensions
  — $S_{q(N,\,\theta)}(p)$ = ROTATION BY ANGLE 2θ ↔ ROTATION IN 3-DIMENSIONS (see Section 6.1)

$T_{q(N,\,\theta)}(p) = q(N, \theta)\, p\, q(N, \theta)$

- Plane ⊥ O, N in four dimensions ≡ plane of vectors ⊥ N in three dimensions
  — $T_{q(N,\,\theta)}(p)$ = IDENTITY ↔ FIXED PLANE IN 3-DIMENSIONS
- Plane of O, N in four dimensions ≡ line through O parallel to N in three dimensions

— $T_{q(N,\,\theta)}(p)$ = rotation by the angle 2θ in 4-dimensions
— $T_{q(N,\pi/2)}(N) = -N \leftrightarrow$ MIRROR IMAGE IN 3-DIMENSIONS (see Section 6.2)
— $T_{q(N,\,\theta)}(N)$ = MASS-POINT $\leftrightarrow$ PERSPECTIVE PROJECTION IN 3-DIMENSIONS (see Section 6.3)

Intuitively, we can see that the sandwiching map $S_{q(N,\,\theta)}$ is the transformation that rotates points or vectors in three dimensions by the angle 2θ around the axis line L passing through the point O in the direction of the unit vector N. Indeed, recall from Section 3.3 that the quaternions in the plane of O, N in four dimensions correspond to mass-points on the axis line L in three dimensions (see Figure 3.6). Moreover, by Corollary 5.2, $S_{q(N,\,\theta)}$ is the identity on the plane of O, N in four dimensions. Therefore the points on the line L in three dimensions are fixed points of the transformation $S_{q(N,\,\theta)}$. In addition, vectors in three dimensions perpendicular to the axis line L correspond to quaternions in four dimensions in the plane perpendicular to O, N, and by Corollary 5.3 $S_{q(N,\,\theta)}$ rotates vectors in this orthogonal plane by the angle 2θ. Therefore in three dimensions $S_{q(N,\,\theta)}$ rotates points and vectors around the axis line L by the angle 2θ. We shall examine this connection between the sandwiching maps $S_{q(N,\,\theta)}$ and rotations in three dimensions in greater detail in Section 6.1.

A linear transformation A on the vector space of mass-points is called an *affine transformation* if A maps points (mass-points with mass = 1) to points and vectors (mass-points with mass = 0) to vectors. Thus a linear transformation A is affine if A preserves mass. The sandwiching transformation $S_{q(N,\,\theta)}$ maps vectors in three dimensions to vectors in three dimensions: $S_{q(N,\,\theta)}(N) = N$ and $S_{q(N,\,\theta)}$ maps vectors in the plane perpendicular to N in three dimensions—that is, quaternions in the plane perpendicular to O, N in four dimensions—to vectors perpendicular to N in three dimensions. Moreover, $S_{q(N,\,\theta)}$ also maps points in three dimensions to points in three dimensions, since $S_{q(N,\,\theta)}(O) = O$, and by linearity for any point P

$$S_{q(N,\,\theta)}(P) = S_{q(N,\,\theta)}(O + (P - O)) = S_{q(N,\,\theta)}(O) + S_{q(N,\,\theta)}(P - O) = O + S_{q(N,\,\theta)}(P - O),$$

which is the sum of a point and a vector and hence a point in three dimensions. Thus the sandwiching maps $S_{q(N,\,\theta)}$ are affine transformations.

In contrast, the sandwiching maps $T_{q(N,\,\theta)}$ are not affine transformations, since by Corollary 5.2, $T_{q(N,\,\theta)}$ rotates quaternions in the plane of O, N by the angle 2θ. Therefore:

$$T_{q(N,\,\theta)}(O) = q(N, 2\theta) = \cos(2\theta)O + \sin(2\theta)N$$

$$T_{q(N,\,\theta)}(N) = \cos(\pi/2 + 2\theta)O + \sin(\pi/2 + 2\theta)N = \sin(2\theta)O + \cos(2\theta)N.$$

Thus, in general, $T_{q(N,\,\theta)}(O)$ is not a point and $T_{q(N,\,\theta)}(N)$ is not a vector; both are mass-points with nonzero mass. Hence the maps $T_{q(N,\,\theta)}$ do not preserve mass.

Linear transformations on the vector space of mass-points that do not preserve mass—that is, that do not map points to points and vectors to vectors—are called *projective transformations*. The most important projective transformations in computer graphics are perspective projections. Perspective projections are the identity on the plane of projection, and the maps $T_{q(N,\,\theta)}$ are the identity on the plane in four dimensions perpendicular to the plane of O, N—that is, the plane in three dimensions of vectors perpendicular to N. These observations suggest that the maps $T_{q(N,\,\theta)}$ may represent perspective projections into planes perpendicular to N. We shall examine the connection between the sandwiching maps $T_{q(N,\,\theta)}$ and perspective projections in more detail in Section 6.3.

The maps $T_{q(N,\,\pi/2)} = T_N$ are rather special. These maps rotate quaternions in the plane of O, N in four dimensions by the angle π. Therefore

$$T_N(O) = -O$$
$$T_N(N) = -N.$$

Since by Corollary 5.3 T_N is the identity on the plane of quaternions in four dimensions perpendicular to the plane of O, N—that is, on the vectors in three dimensions perpendicular to N—and $T_N(N) = -N$, it follows that in three dimensions the transformation T_N maps vectors to vectors, but not points to points since $T_N(O) = -O$ has mass $= -1$.

We shall call transformations that map vectors to vectors, but not points to points—that is, transformations that preserve zero mass—*semi-affine transformations*. The maps T_N are semi-affine transformations. These transformations behave well on vectors in three dimensions but may not behave as expected on points in three dimensions. Since the map T_N is the identity on the vectors perpendicular to N and $T_N(N) = -N$, the map T_N computes the mirror image of vectors in three dimensions in the plane perpendicular to the vector N. The effect of the map T_N on points in three dimensions, however, is not so obvious. We shall examine the connection between the sandwiching maps T_N and reflections in further detail in Section 6.2.

6.1 ROTATION

Suppose we want to rotate a point P or a vector v by the angle θ around an axis line L passing through a point O in the direction of a unit vector N (see Figure 6.1). To compute the effect of this rotation on the vector v, we shall decompose v into two components $v_{||}$ and $v_{\perp}$ such that

$$v = v_{||} + v_{\perp},$$

where

$v_{||}$ = component of v parallel to N

$v_{\perp}$ = component of v perpendicular to N

(see Figure 6.2 a).

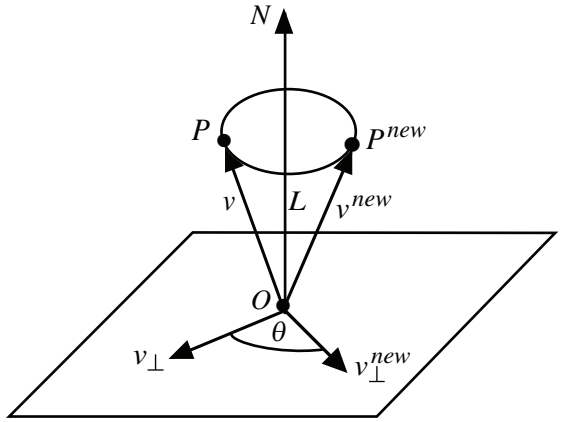
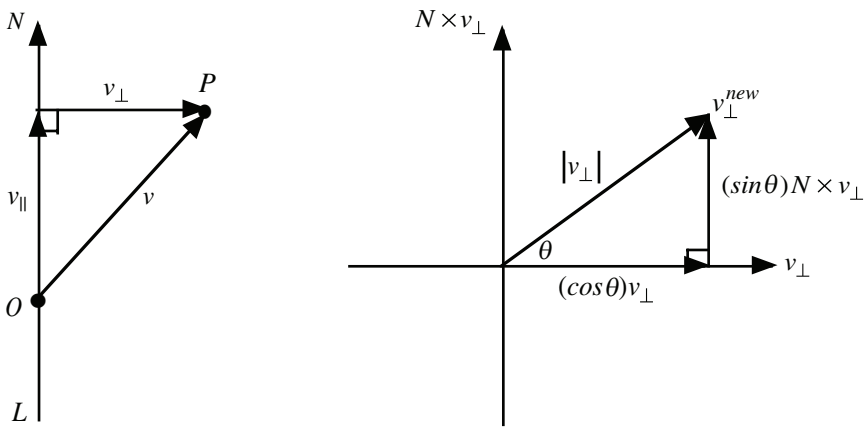


FIGURE 6.1: Rotating a point P or a vector v by the angle θ around the axis line L through the point O in the direction of the vector N.

Since $v_{||}$ is parallel to the axis of rotation N, $v_{||}$ is not altered by rotation. Hence after rotation

$$v_{||}^{new} = v_{||}.$$

Thus to compute the effect of rotation on v, we need only compute the effect of rotation on $v_{\perp}$. Now $v_{\perp}$ lies in the plane perpendicular to the axis vector N; moreover, $N \times v_{\perp}$ is perpendicular to $v_{\perp}$ and also lies in the plane perpendicular to N (see Figure 6.2 b). Hence after rotation



(a) $v = v_{||} + v_{\perp}$ (b) $v_{\perp}^{new} = (\cos\theta)v_{\perp} + (\sin\theta)N \times v_{\perp}$

FIGURE 6.2: Decomposing a vector v into components parallel ($v_{||}$)and perpendicular ($v_{\perp}$) to the axis of rotation (left) and rotation in the plane perpendicular to the axis of rotation (right).

$$v_{\|}^{new} = (\cos\theta)v_{\perp} + (\sin\theta)N \times v_{\perp}.$$

Therefore, by linearity, after rotation

$$v^{new} = v_{\|}^{new} + v_{\perp}^{new} = v_{\|} + (\cos\theta)v_{\perp} + (\sin\theta)N \times v_{\perp}. \tag{6.1}$$

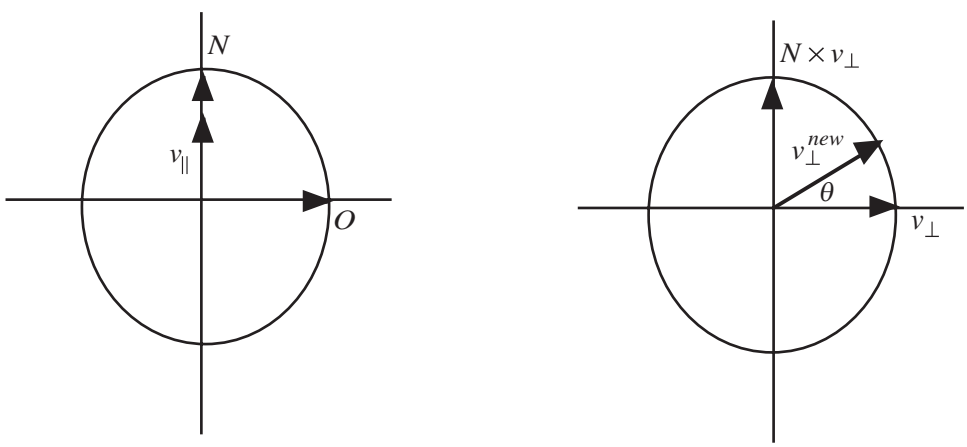
Similarly for points, since $P = O + (P - O)$, it follows by linearity that after rotation

$$P^{new} = O^{new} + (P - O)^{new} = O + (P - O)^{new}. \tag{6.2}$$

Figures 6.1 and 6.2 describe rotation around the axis line L in three dimensions. Now let us look at the analogous pictures in the 4-dimensional space of quaternions. In the space of quaternions, the line L through the point O in the direction of the vector N is represented by the plane of O, N, since every quaternion (mass-point)

$$q = c\,O + s\,N \equiv O + \frac{s}{c}N$$

in this plane in four dimensions lies on the line L in three dimensions (see Figure 3.6). In contrast, the plane of vectors in three dimensions perpendicular to the vector N is represented by the complementary plane of quaternions in four dimensions perpendicular to the plane of O, N (see Figure 3.5). These results are summarized in Figure 6.3.



(a) the plane of O, N *(b) the plane perpendicular to O, N*

FIGURE 6.3: (a) The plane O, N of quaternions (mass-points) in four dimensions represents the line L in three dimensions passing through the point O in the direction of the unit vector N. (b) The plane of quaternions perpendicular to O, N in four dimensions represents the vectors in three dimensions orthogonal to the unit vector N. Compare with Figures 3.5, 3.6, and 6.2.

To rotate vectors in three dimensions around the axis line L using quaternion multiplication, we need to keep the plane of O, N fixed (since the axis line L is fixed by rotation around L) and to rotate quaternions in the plane perpendicular to O, N (since these vectors are perpendicular to the axis vector N in three dimensions) by the angle θ. But by Corollaries 5.2 and 5.3 these results are precisely the effects of the sandwiching map $S_{q(N,\theta/2)}$. Thus we are led to the following theorem.

Theorem 6.1: *Sandwiching with conjugates rotates vector in 3-dimensional*

Let

- N = *a unit vector*
- v = *a vector in three dimensions* (*i.e.*, *a quaternion* $\perp O$)

Then

- $S_{q(N,\theta/2)}(v) = q(N,\theta/2)v\, q^*\,(N,\theta/2)$ *rotates* v *by the angle* θ *around the axis* N.

Proof: Let $v = v_{\|} + v_{\perp}$, where

- $v_{\|}$ = component of v parallel to N
- $v_{\perp}$ = component of v perpendicular to N.

Then

- $S_{q(N,\theta/2)}(v_{\|}) = q(N,\theta/2)\, v_{\|}\, q^*(N,\theta/2) = v_{\|}$ (Corollary 5.2)
- $S_{q(N,\theta/2)}(v_{\perp}) = q(N,\theta/2)\, v_{\perp}\, q^*(N,\theta/2) = \cos(\theta)v_{\perp} + \sin(\theta)N \times v_{\perp}$
  rotates $v_{\perp}$ *by the angle* θ *in the plane* $\perp$ *to* O, N. (Corollary 5.3)

Therefore since $S_{q(N,\theta/2)}$ is a linear transformation and $v = v_{||} + v_{\perp}$,

$$S_{q(N,\theta/2)}(v) = S_{q(N,\theta/2)}(v_{||}) + S_{q(N,\theta/2)}(v_{\perp}) = v_{||} + (\cos\theta)v_{\perp} + (\sin\theta)N \times v_{\perp}.$$

Hence by Equation (6.1) sandwiching has the same effect on v as rotating v in three dimensions by the angle θ around the axis vector N. ♦

The sandwiching map $S_{q(N,\theta/2)}$ also rotates affine points in three dimensions about the line L passing through the origin O parallel to the vector N. Indeed by Corollary 5.2, the map $S_{q(N,\theta/2)}$ is the identity on the quaternion plane in four dimensions determined by O, N. Therefore the points on the line L are fixed points of the transformation $S_{q(N,\theta/2)}$. Thus the sandwiching map $S_{q(N,\theta/2)}$ rotates points as well as vectors in three dimensions.

Theorem 6.2: *Sandwiching with conjugates also rotates points in 3-dimensional*

Let

- N = *a unit vector*
- P = *a point in three dimensions*

Then

- $S_{q(N,\theta/2)}(P) = q(N,\theta/2)\ P q^*(N,\theta/2)$ *rotates P by the angle θ around the line L through the origin O parallel to the vector N.*

Proof: Since P is a point in affine space,

$$P = O + (P - O).$$

But

- $S_{q(N,\theta/2)}(O) = O$ (Corollary 5.2)
- $S_{q(N,\theta/2)}(P - O) =$ *rotates the vector $P - O$ by the angle θ around the axis N* (Theorem 6.1)

Therefore since $S_{q(N,\theta/2)}$ is a linear transformation and $P = O + (P - O)$,

$$S_{q(N,\theta/2)}(P) = O + S_{q(N,\theta/2)}(P - O).$$

Hence by Theorem 6.1 and Equation (6.2) sandwiching has the same effect on P as rotating P in three dimensions by the angle θ around the line L passing through the origin O parallel to the vector N. ♦

We close this section with the following well-known result about composing rotations using quaternion multiplication.

Proposition 6.3: $(S_q \circ S_r)\ (p) = S_{qr}(p)$

Therefore composing two rotations is equivalent to multiplying two unit quaternions.

Proof: By Proposition 4.4

$$(S_q \circ S_r)\ (p) = S_q(S_r(p)) = S_q(rpr^*) = qrpr^*q^* = (qr)\ p\ (qr)^* = S_{qr}(p).$$

Exercises:

6.1. Let q be an arbitrary quaternion. Define

$$e^q = O + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \cdots$$

a. Show that if N is a unit vector, then

$$e^{\lambda N} = cos(\lambda)\mathrm{O} + sin(\lambda)N.$$

b. Conclude from part a and Theorem 6.1 that a vector v can be rotated around a unit vector N through the angle ϕ by sandwiching v between $q = e^{\phi N/2}$ and $q^* = e^{-\phi N/2}$.

6.2 In this exercise, we will study the composite of two rotations by examining the product of two unit quaternions.

a. Show that that if p and q are unit quaternions, then pq is a unit quaternion.

b. Using part a, show that

$$q(N_1, \phi_1)q(N_2, \phi_2) = q(N, \phi),$$

where

i. $\cos(\phi/2) = \cos(\phi_1/2)\cos(\phi_2/2) - \sin(\phi_1/2)\sin(\phi_2/2)(N_1 \cdot N_2)$

ii. $\sin(\phi/2)N = \cos(\phi_2/2)\sin(\phi_1/2)N_1 + \cos(\phi_1/2)\sin(\phi_2/2)N_2 + \sin(\phi_1/2)\sin(\phi_2/2)N_1 \times N_2$

c. Conclude from part b that

$$q(N, \phi_1)\, q(N, \phi_2) = q(N, \phi_1 + \phi_2) \quad .$$

6.2 MIRROR IMAGE

Suppose we want to mirror a point P or a vector v in a plane S through a point O perpendicular to a unit vector N (see Figure 6.4, left). To compute the effect of this reflection on the vector v, we decompose v into two components $v_{\|}$ and $v_{\perp}$ such that

$$v = v_{\|} + v_{\perp},$$

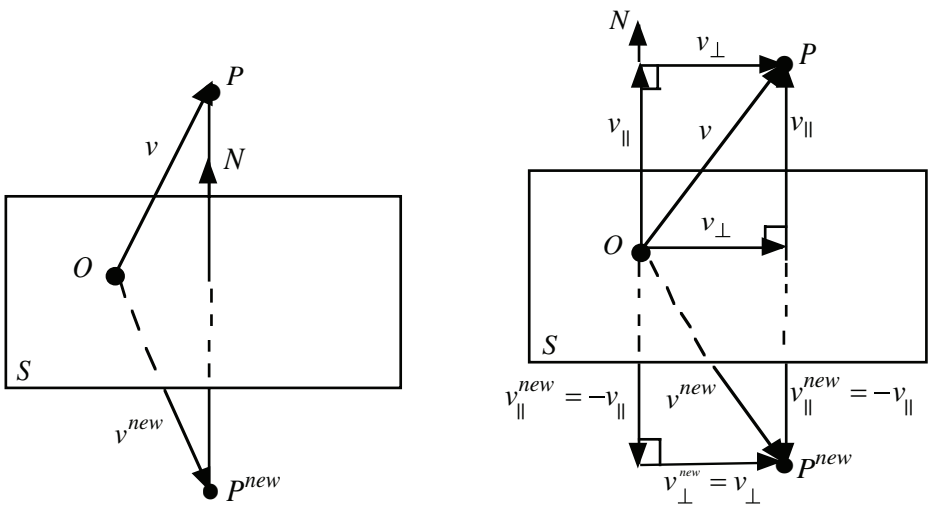


FIGURE 6.4: A mirror plane S specified by a point O and a unit normal vector N (left) and the decomposition of a vector v into components parallel and perpendicular to the normal vector (right).

where as usual

$v_{\|}$ = component of v parallel to N

$v_{\perp}$ = component of v perpendicular to N.

(see Figure 6.4, right).

Since $v_{\perp}$ is perpendicular to the normal N, the vector $v_{\perp}$ is not altered by reflection. Hence after reflection

$$v_{\perp}^{new} = v_{\perp}.$$

Thus to compute the effect of reflection on v, we need only compute the effect of reflection on $v_{\|}$. But reflection simply reverses the direction of $v_{\|}$, so

$$v_{\|}^{new} = -v_{\|}.$$

Therefore, by linearity, after reflection

$$v^{new} = v_{\perp} - v_{\|}. \tag{6.3}$$

Similarly for points, since $P = O + (P - O)$, it follows by linearity that after reflection

$$P^{new} = O + (P - O)^{new}. \tag{6.4}$$

To use sandwiching to compute the mirror image of a vector v in a plane perpendicular to N, notice that by Corollary 5.3 the maps $T_{q(N,\theta)}$ are the identity on vectors perpendicular to N. Therefore, by Equation (6.3), to compute mirror image in a plane perpendicular to N, we need only find an angle θ for which $T_{q(N,\theta)}$ maps N to $-N$. Since by Corollary 5.2 the map $T_{q(N,\theta)}$ rotates quaternions in the plane of O, N by the angle 2θ, the angle we seek is $\theta = \pi/2$. But

$$q(N, \pi/2) = \cos(\pi/2)O + \sin(\pi/2)N = N.$$

Therefore we are led directly to the following result.

Theorem 6.4: *Sandwiching with unit vectors reflects vectors in 3-dimensional*

Let

- N = *a unit vector*
- v = *a vector in three dimensions* (*i.e.*, *a quaternion* $\perp O$)

Then

- $T_N(v) = NvN = -S_N(v)$ *is the mirror image of* v *in the plane* $\perp N$.

Proof: Let $v = v_{\|} + v_{\perp}$, where

- $v_{\|}$ = component of v parallel to N
- $v_{\perp}$ = component of v perpendicular to N.

Since $N = \cos(\pi/2)O + \sin(\pi/2)N = q(\mathrm{N}, \pi/2)$:

- $T_N(v_{\|}) = N v_{\|} N = q(N, \pi/2)\, v_{\|}\, q(N, \pi/2) = T_{q(N,\pi/2)}(v_{\|}) = -v_{\|}$ (Corollary 5.2)
- $T_N(v_{\perp}) = N v_{\perp} N = q(N, \pi/2)\, v_{\perp}\, q(N, \pi/2) = T_{q(N,\pi/2)}(v_{\perp}) = v_{\perp}$ (Corollary 5.3)

Therefore, since T_N is a linear transformation and $v = v_{\perp} + v_{\|}$,

$$T_N(v) = T_N(v_{\perp}) + T_N(v_{\|}) = v_{\perp} - v_{\|}$$

Hence by Equation (6.3) sandwiching with N has the same effect on v in three dimensions as reflecting v in a plane perpendicular to N. ♦

Proposition 6.5: $(T_{N_2} \circ T_{N_1})(v) = S_{N_2 N_1}(v)$
Therefore the composite of two reflections is equivalent to one rotation.
Proof: By Equation (4.6)

$$N_2 N_1 = (-N_2 \cdot N_1)O + N_2 \times N_1$$

$$N_1 N_2 = (-N_1 \cdot N_2)O + N_1 \times N_2 = (-N_2 \cdot N_1)O - N_2 \times N_1.$$

so

$$N_1 N_2 = (N_2 N_1)^*.$$

Therefore

$$(N_2 N_1)\, v\, (N_1 N_2) = (N_2 N_1)\, v\, (N_2 N_1)^*,$$

so

$$(T_{N_2} \circ T_{N_1})(v) = (S_{N_2 N_1})(v). \quad ♦$$

In fact, every rotation on vectors in three dimensions is the composite of two reflections. For let u, v be two unit vectors in the plane perpendicular to N, chosen so that the vectors u, v, N have positive orientation and the angle between u and v is $\theta/2$. Then by Equation (4.6)

$$-v\,u = (v \cdot u)O - v \times u = \cos(\theta/2)O + \sin(\theta/2)N = q(N, \theta/2).$$

Therefore

$$S_{q(N,\theta/2)}(w) = (-v\,u)\; w\; (-u\,v) = (T_v \circ T_u)(w).$$

Thus the composite of two reflections $(T_v \circ T_u)$ is equivalent to a single rotation S_{-vu}, and every rotation $S_{q(N,\theta/2)}$ is the composite of two reflections.

As a consequence of Theorems 6.1 and 6.4, every rigid motion on vectors in three dimensions—every rotation or reflection—is the composite of two rotations in four dimensions: one rotation represented by multiplication with a unit quaternion on the left and the other rotation represented by multiplication with a unit quaternion on the right. The trick is simply to balance these rotations in the planes parallel and perpendicular to O,N. In the plane parallel to O,N, multiplying by $q(N,\theta)$ on both the left and the right represents counterclockwise rotation by the angle θ, but in the plane perpendicular to O,N, multiplying by $q(N,\theta)$ on the left represents counterclockwise rotation by the angle θ, whereas multiplying by $q(N,\theta)$ on the right represents clockwise rotation by the angle θ. Thus these rotation angles add in the plane parallel to O,N, and subtract in the plane perpendicular to O,N. We simply take advantage of these additions and subtractions to cancel the rotation in the appropriate plane in four dimensions to produce either rotation around the axis vector N or mirror image in the plane perpendicular to the vector N in three dimensions.

Since by Theorem 6.2 the sandwiching maps S_q can be used to rotate points as well as vectors in three dimensions around lines through the origin, Theorem 6.4 seems to invite us to use the sandwiching maps T_N to compute the mirror image of points in planes through the origin. But recall that unlike the transformations S_q, the maps T_N are not affine maps; the transformations T_N are only semi-affine. The function

$$T_N(p)=N\,p\,N$$

is not the identity on the quaternion plane in four dimensions determined by O, N. Therefore even though the map $T_N(v)$ is the identity on the vectors v perpendicular to N in three dimensions, we find that

$$T_N(O)=N\,O\,N=-O\neq O.$$

Hence $T_N(P)$ is not the identity on the affine plane in three dimensions through the point O perpendicular to the vector N. Thus, perhaps contrary to intuition, sandwiching a point P with the unit vector N is not the mirror image in three dimensions of the point P in the plane through the origin O perpendicular to the vector N. Rather, we have the following somewhat disappointing result.

Theorem 6.6: *Sandwiching with N rotates points through the angle π in 3-dimensions*

Let

- *N = a unit vector*
- *P = a point in three dimensions*

Then

- $T_N(P) = N\,P\,N$ *rotates* P *by the angle* π *around the line passing through the origin* O *parallel to the vector* N.

Proof: Since P is a point in affine space,

$$P = O + (P - O).$$

But

- $T_N(O) = N\,O\,N = N^2 = -O$
- $T_N(P - O) = N(P - O)\,N$

Therefore since T_N is a linear transformation and $P = O + (P - O)$,

$$T_N(P) = -O + N(P - O)N \equiv O - N(P - O)\,N = O + N(P - O)\,N^*.$$

Hence

$$T_N(P) \equiv S_N(P).$$

Since $N = \cos(\pi/2)O + \sin(\pi/2)N = q(N, \pi/2)$, it follows by Theorem 6.2 that $T_N(P)$ rotates P by the angle π around the line passing through the origin O parallel to the vector N. ♦

Thus sandwiching with N does not reflect points P in the plane through the origin O perpendicular to the unit vector N. Nevertheless we can compute the mirror image of points in this plane by sandwiching using the following approach.

Theorem 6.7: *Sandwiching* $P - 2O$ *with* N *reflects* P *in the plane through* $O \perp N$

Let

- N = *a unit vector*
- P = *a point in three dimensions*

Then

- $T_N(P - 2O) = N\,(P - 2O)N$ *is the mirror image of the point* P *in the plane through* O *perpendicular to the unit vector* N.

Proof: Clearly

$$P - 2O = (P - O) - O.$$

But

- $T_N(-O) = -N\,O\,N = -N^2 = O$
- $T_N(P - O) = N(P - O)N$

Therefore since T_N is a linear transformation and $P-2O=(P-O)-O$,

$$T_N(P-2O)=O+N(P-O)N,$$

which by Theorem 6.4 and Equation (6.4) is the mirror image of the point P in the plane through O orthogonal to the unit normal vector N. ♦

If we introduce rectangular coordinates, then $P=(p_1,p_2,p_3,1)$ and $O=(0,0,0,1)$, so

$$P-2O=(p_1,p_2,p_3,-1).$$

Thus $P-2O$ is the mass-point with mass -1 located at the point with coordinates $(-p_1,-p_2,-p_3)$. By Theorem 6.7, to find the mirror image of a point P in the plane through O orthogonal to the unit vector N, we simply negate the fourth coordinate, the mass, of P and sandwich the resulting mass-point with N.

Exercise:

6.3. Let v,w be unit vectors.

a. Show that

i. vw is a unit quaternion.

ii. $S_{vw}(w)$ lies in the plane of the vectors v,w.

iii. v bisects the angle between w and $S_{vw}(w)$.

b. Prove that mirroring a vector u in the plane perpendicular to w and then mirroring the resulting vector in the plane perpendicular to v is equivalent to rotating the vector u around the vector $v\times w$ by twice the angle between w and v.

6.3 PERSPECTIVE PROJECTION

We have examined the maps $S_{q(N,\theta)}(v)$ that sandwich the vector v between the unit quaternions $q(N,\theta)$ and $q^*(N,\theta)$ for all unit quaternions

$$q(N,\theta)=\cos(\theta)O+\sin(\theta)N,$$

and we have seen in Theorem 6.1 that in all cases $S_{q(N,\theta)}(v)$ rotates the vector v around the unit vector N by the angle 2θ. But so far we have investigated the maps $T_{q(N,\theta)}(v)$ that sandwich the vector v between two copies of the unit quaternion $q(N,\theta)$ only when $q(N,\theta)=N$—that is, only for $\theta=\pi/2$. By Theorem 6.4, $T_N(v)$ is the mirror image of the vector v in the plane perpendicular to the unit vector N. We are now going to study the maps $T_{q(N,\theta)}$ for $\theta\neq\pi/2$.

By Corollary 5.3, $T_{q(N,\theta)}$ is the identity on the plane of vectors in three dimensions perpendicular to N. Therefore we might expect that the map $T_{q(N,\theta)}$ represents some kind of projection into the

plane perpendicular to N. On the other hand, projections in three dimensions collapse a dimension, but we know that in four dimensions the map $T_{q(N,\theta)}$ is an isometry, so it is unclear how an isometry in four dimensions can represent a projection in three dimensions. Nevertheless, here we shall show that the maps $T_{q(N,\theta)}$ can indeed be used to compute 3-dimensional perspective projections. To understand how this works, we shall begin by developing explicit formulas for perspective projection.

6.3.1 Perspective Projection and Singular 4 × 4 Matrices

Suppose we want to project a point P from an eye point E into a plane S passing through a point Q and perpendicular to a unit vector N (see Figure 6.5). To compute the effect of this projection on the point P, observe that by similar triangles the perspective projection of P from E onto S is given by

$$P^{new} = E + r(P - E)$$

where

$$r = \frac{|P^{new} - E|}{|P - E|} = \frac{(P^{new} - E)\cdot N}{(P - E)\cdot N} = \frac{(Q - E)\cdot N}{(P - E)\cdot N}.$$

Substituting this expression for r into the equation for P^{new} yields

$$P^{new} = E + \frac{(Q - E)\cdot N}{(P - E)\cdot N}(P - E) = \frac{\big((P - E)\cdot N\big)E + \big((Q - E)\cdot N\big)(P - E)}{(P - E)\cdot N}. \tag{6.5}$$

or equivalently

$$P^{new} = \frac{\big((P - Q)\cdot N\big)E + \big((Q - E)\cdot N\big)P}{(P - E)\cdot N}. \tag{6.6}$$

Treating the denominator as mass, we find that

$$P^{new} = ((P - Q)\cdot N)E + (Q - E)\cdot N)P,\ (P - E)\cdot N) \tag{6.7}$$

is the mass-point, where

- the point is located at the perspective projection of the point P from the eye point E onto the plane S;
- the mass is equal to the signed distance of the point P from the plane through the eye point E perpendicular to the unit normal N, i.e., the plane through the eye point E parallel to the plane of projection S.

Notice that since the mass $(P - E)\cdot N$ of the mass-point P^{new} is the signed distance from the point P to the plane through the eye point E parallel to the plane of projection S, we can use this formula for perspective projection to detect hidden surfaces: if two points project to the same point, then the smaller the mass, the closer the surface point is to the eye. Points further from the plane of the eye are hidden by points closer to the plane of the eye.

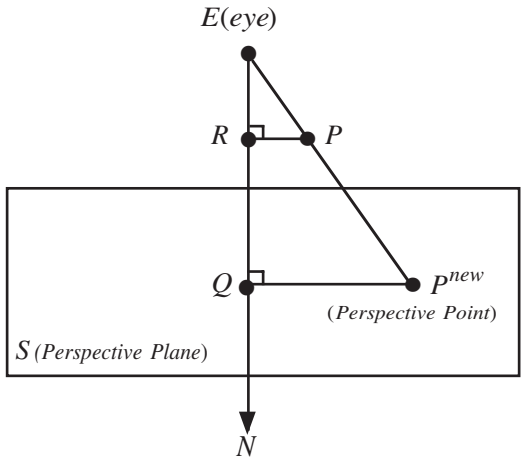


FIGURE 6.5: Perspective projection from the eye point E into the plane S through the point Q with unit normal N. The point P is projected to the point P^{new}, located at the intersection of the line EP with the plane S. The triangles ΔERP and ΔEQP^{new} are similar, so $\frac{|P^{new}-Q|}{|Q-E|}=\frac{|P-R|}{|R-E|}$.

We can easily find a 4×4 matrix to represent this transformation. Let I denote the 3×3 identity matrix and let N^T denote the transpose of N—that is, the column vector with the same entries as the row vector N. Then it follows directly from Equation (6.7) and matrix multiplication that

$$P^{new}=(P,1)*\begin{pmatrix}((Q-E)\cdot N)I+N^T*E & N^T\\ -(Q\cdot N)E & -E\cdot N\end{pmatrix},$$

Thus the 4×4 matrix $Persp(E,Q,N)$ representing perspective projection from the eye point E to the plane S passing through the point Q and perpendicular to the unit vector N is given by

$$Persp(E,Q,N)=\begin{pmatrix}((Q-E)\cdot N)I+N^T*E & N^T\\ -(Q\cdot N)E & -E\cdot N\end{pmatrix} \tag{6.8}$$

The 4×4 matrix $Persp(E,Q,N)$ is singular, since

$$(E,1)*Persp(E,Q,N)=(E,1)*\begin{pmatrix}((Q-E)\cdot N)I+N^T*E & N^T\\ -(Q\cdot N)E & -E\cdot N\end{pmatrix}=(0,0,0,0) \tag{6.9}$$

Thus the rows of $Persp(E,Q,N)$ are linearly dependent.

Notice, however, that by linearity and Equation (6.9)

$$\begin{aligned}(P-E,0)*\,Persp(E,Q,N) &= (P,1)*Persp(E,Q,N)-(E,1)*Persp(E,Q,N)\\ &= (P,1)*Persp(E,Q,N)\end{aligned} \tag{6.10}$$

Thus we can apply the matrix $Persp(E,Q,N)$ directly to the vector $(P-E,0)$, instead of to the point $(P,1)$. Applying $Persp(E,Q,N)$ to vectors $(P-E,0)$ is more efficient than applying $Persp(E,Q,N)$ to points $(P,1)$ because the zero in the fourth component of $(P-E,0)$ allows us to ignore the fourth row of $Persp(E,Q,N)$. Thus we can replace the 4×4 matrix $Persp(E,Q,N)$ by the upper 3×4 submatrix of $Persp(E,Q,N)$ saving both memory and computation. Moreover, we shall see shortly (Examples 6.1 and 6.2 below) that we can often choose the fourth row so that $Persp(E,Q,N)$ is an orthogonal matrix and hence a rotation in four dimensions. (For a more complete discussion, see Section 8.4.)

6.3.2 Perspective Projection by Sandwiching with Quaternions

Now back to quaternions. Here we are going to explore alternative ways of computing perspective projection by using sandwiching with quaternions $q(N,\theta)$ to operate on the vectors $P-E$.

To simplify our initial discussion, we shall start with the special case where $\theta=-\pi/4$. By Corollary 5.3 we know that for all θ, the map $T_{q(N,\theta)}$ is the identity on vectors in the plane of quaternions in four dimensions perpendicular to O, N, and from Corollary 5.2 we know that the map $T_{q(N,-\pi/4)}$ rotates quaternions in the plane in four dimensions determined by O, N clockwise by the angle $2(-\pi/4)=-\pi/2$. In particular, $T_{q(N,-\pi/4)}$ maps the unit vector N to the point O and the point O to the unit vector $-N$ (see Figure 6.6). Thus $T_{q(N,-\pi/4)}$ converts distance along the unit vector N into mass at the point O. We shall now show how to take advantage of this property of $T_{q(N,-\pi/4)}$ to compute perspective projection. As a bonus, we shall see that we can also use this sandwiching formula for perspective projection to detect hidden surfaces. Moreover, since $T_{q(N,-\pi/4)}$ represents a

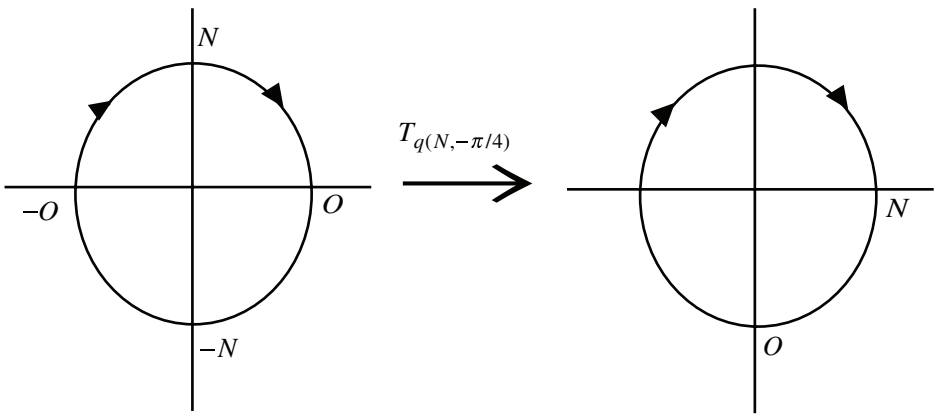


FIGURE 6.6: $T_{q(N,-\pi/4)}$ maps the unit vector N to the point O and the point O to the unit vector $-N$.

rotation in four dimensions, this sandwiching formula will allow us to use 4×4 orthogonal matrices to compute perspective projections in three dimensions.

Theorem 6 .8: *Sandwiching vectors to the eye with $q(N,-\pi/4)$ gives perspective*

Let

- *S = the plane through the origin O perpendicular to the unit normal N*
- *$E = O-N$ eye point*
- *P = a point in three dimensions*

Then

- *$T_{q(N,-\pi/4)}(P-E)=q(N,-\pi/4)\ (P-E)\ q(N,-\pi/4)$ is a mass-point, where*
- – *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*
- – *the mass is equal to the distance d of the point P from the plane through the eye point E perpendicular to the unit normal N.*

Proof: Let $P-E=dN+v$, where $v\perp N$ (see Figure 6.7). Then

- $T_{q(N,-\pi/4)}(dN)=dO$ (Corollary 5.2)
- $T_{q(N,-\pi/4)}(v)=v.$ (Corollary 5.3)

Hence by linearity

- $T_{q(N,-\pi/4)}(P-E)=T_{q(N,-\pi/4)}(dN+v)=dO+v\equiv O+v/d.$

Therefore, by similar triangles (see Figure 6.7), the point $O+v/d$ corresponding to the mass-point $T_{q(N,-\pi/4)}(P-E)$ is the perspective projection of the point P from the eye point E onto the plane S.

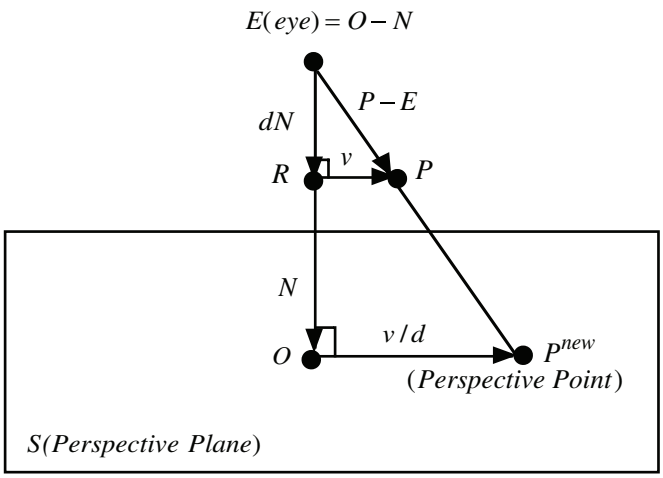


FIGURE 6.7: By similar triangles, the point $P^{new}=O+v/d$ is the intersection of the line joining the eye point E to the point P with the plane S through the origin O perpendicular to the unit normal N.

Moreover, the mass d is equal to the distance of the point P from the plane through the eye point E perpendicular to the unit normal N. ♦

Notice that since the mass d of the mass-point $T_{q(N,-\pi/4)}(P-E)$ that we compute with sandwiching is the distance from the point P to the plane through the eye point E parallel to the plane of projection S, we can use this sandwiching formula for perspective projection to detect hidden surfaces: if two points project to the same point, then the smaller the mass, the closer the surface point is to the eye. Points further from the plane of the eye are hidden by points closer to the plane of the eye.

In the following example we show how this sandwiching formula for perspective projection allows us to use 4×4 orthogonal matrices to compute perspective projection.

Example 6.1 4×4 *Orthogonal Matrices and Perspective Projections*
Suppose that the eye is located along the z-axis at the point $E=(0,0,1)$ and the perspective plane S is the xy-plane with unit normal $N=(0,0,-1)$. Thus the plane S passes through the origin $Q=(0,0,0)$. For any point $P=(x,y,z)$, it follows by Equation (6.5) that the line EP intersects the plane S at the point

$$P^{new}=\frac{\big((P-E)\cdot N\big)E+\big((Q-E)\cdot N\big)(P-E)}{(P-E)\cdot N}=\left(\frac{x}{1-z},\frac{y}{1-z},0\right)$$

By Equation (6.8) the standard singular 4×4 matrix $Persp(E,Q,N)$ representing this perspective projection is given by

$$Persp(E,Q,N)=\begin{pmatrix}\big((Q-E)\cdot N\big)I+N^T*E & N^T\\ -(Q\cdot N)E & -E\cdot N\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&0&1\end{pmatrix},$$

and we can easily verify that

$$\begin{aligned}(P-E,0)*Persp(E,Q,N)&=(x,y,z-1,0)*\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&0&1\end{pmatrix}\\&=(x,y,0,1-z)\equiv\left(\frac{x}{1-z},\frac{y}{1-z},0\right)\end{aligned}$$

Since the map $T_{q(N,-\pi/4)}$ represents a rotation in four dimensions, it follows by Theorem 6.8 that there is a 4×4 orthogonal matrix R corresponding to the sandwiching map $T_{q(N,-\pi/4)}$ such that

$$(P-E,0)*R=(P-E,0)*Persp(E,Q,N).$$

The map $T_{q(N,-\pi/4)}$ represents a rotation in four dimensions in the O, N plane. Since the plane of projection is the xy-plane and since the unit normal N points along the negative z-axis, the plane of O, N in four dimensions is the $w(-z)$-plane. Thus in four dimensions the sandwiching map $T_{q(N,-\pi/4)}$ is just rotation by $-\pi/2$ in the $w(-z)$-plane or equivalently rotation by $-\pi/2$ in the zw-plane. Therefore the 4×4 orthogonal matrix R corresponding to the map $T_{q(N,-\pi/4)}$ is given by

$$R=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

To check that R correctly performs perspective projection, observe that

$$(P-E,0)*R=(x,y,z-1,0)*\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}=(x,y,0,1-z)\equiv\left(\frac{x}{1-z},\frac{y}{1-z},0\right)$$

as required.

Perspective projection is not an affine transformation; perspective projection does not preserve mass. Therefore perspective can map vectors to mass-points, and in Theorem 6.8 vectors are indeed mapped to mass-points: the vector from the eye point E to the point P is mapped to the mass-point P^{new} located at the perspective projection of the point P from the eye point E to the plane S.

One could also ask what is the effect of applying the sandwiching map $T_{q(N,-\pi/4)}$ directly to points P: does $T_{q(N,-\pi/4)}(P)$ also compute the perspective projection of P onto the plane S? Alas no. The sandwiching map $T_{q(N,-\pi/4)}$ transforms the origin O to the vector $-N$, so even for the origin the map $T_{q(N,-\pi/4)}$ does not compute perspective projection. The map $T_{q(N,-\pi/4)}$ is not affine, so we cannot expect the map $T_{q(N,-\pi/4)}$ to be well-behaved both on points and on vectors. Much like the mirror image maps T_N, we only get the desired result for vectors, not for points. Even though $P=E+(P-E)$, the identity

$$T_{q(N,-\pi/4)}(P)=T_{q(N,-\pi/4)}(E)+T_{q(N,-\pi/4)}(P-E)$$

does not really help, since $T_{q(N,-\pi/4)}(E)$ is not a point, but is rather a mass-point. Notice, however, that

$$T_{q(N,-\pi/4)}(P)-T_{q(N,-\pi/4)}(E)=T_{q(N,-\pi/4)}(P-E),$$

so subtracting the constant mass-point $T_{q(N,-\pi/4)}(E)$ from the mass-point $T_{q(N,-\pi/4)}(P)$ computed by sandwiching does indeed give perspective projection.

Theorem 6.8 is a special case of the following more general result.

Theorem 6.9: *Sandwiching vectors to the eye with $q(N,-\theta/2)$ gives perspective*

Suppose that $0<\theta<\pi$ and *let*

- S = *the plane through the point* $O+\cot(\theta)N \equiv T_{q(N,-\theta/2)}(N)$ *perpendicular to the unit normal* N
- $E=O+(\cot(\theta)-\csc(\theta))N$ = *eye point*
- P = *a point in three dimensions*

Then

- $T_{q(N,-\theta/2)}(P-E)=q(N,-\theta/2)(P-E)\,q(N,-\theta/2)$ *is a mass-point, where*
- – *the point is located at the perspective projection of the point P from the eye point E onto the plane S;*
- – *the mass is equal to $d\sin(\theta)$, where d is the distance of the point P from the plane through the eye point E perpendicular to the unit normal N.*

Proof: Once again let $P-E=dN+v$, where $v\perp N$ (see Figure 6.8). Then

- $T_{q(N,-\theta/2)}(dN)=d\cos(\pi/2-\theta)O+d\sin(\pi/2-\theta)N=d\sin(\theta)O+d\cos(\theta)N$ (Corollary 5.2)
- $T_{q(N,-\theta/2)}(v)=v$ (Corollary 5.3)

Therefore by linearity

$$T_{q(N,-\theta/2)}(P-E)=T_{q(N,-\theta/2)}(dN+v)=d\sin(\theta)O+d\cos(\theta)N+v$$
$$\equiv O+\cot(\theta)N+\csc(\theta)\frac{v}{d}\,.$$

Since by construction

- $O+\cot(\theta)N=E+\csc(\theta)N$,

it follows again by similar triangles (see Figure 6.8) that the point corresponding to the mass-point $T_{q(N,-\theta/2)}(P-E)$ is the perspective projection of the point P from the eye point E onto the plane S. Moreover, the mass is $d\sin(\theta)$, where d is the distance of the point P from the plane through the eye point E perpendicular to the unit normal N. ♦

By Theorem 6.9, the maps $T_{q(N,-\theta/2)}(v)$ can be used to compute perspective projection onto a plane perpendicular to the unit vector N. The angle θ controls the distance $d=|\csc(\theta)|\geq 1$ from

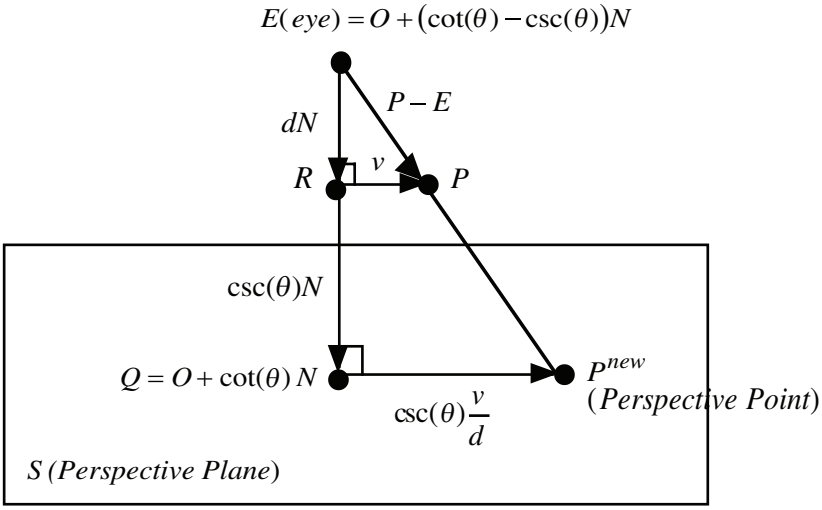


FIGURE 6.8: By similar triangles, the point

$$P^{new} = O + \cot(\theta)\,N + \csc(\theta)\frac{v}{d} = E + \csc(\theta)N + \csc(\theta)\frac{v}{d}$$

is the perspective projection of the point P from the eye point E onto the plane S through the point $Q = O + \cot(\theta)\,N$ perpendicular to the unit normal N.

the eye point E to the plane of projection. When $\theta = \pi$, the eye recedes to infinity and the map $T_{q(N,-\pi/2)}(v) = T_{-N}(v)$ computes the mirror image of the vector v in the plane perpendicular to N.

Notice again that since the mass of the mass-point that we compute with sandwiching is a constant ($\sin(\theta)$) times the distance d from the point P to the plane through the eye E parallel to the plane of projection S, we can still use this sandwiching formula for perspective projection to detect hidden surfaces: once again if two points project to the same point, then the smaller the mass, the closer the surface point is to the eye. Thus even though we are projecting a point onto a plane in three dimensions, no information is lost: distance is simply converted into mass.

This sandwiching formula for perspective projection again allows us to use 4×4 orthogonal matrices to compute perspective projection, this time when the eye is at an arbitrary distance from the perspective plane.

Example 6.2 4×4 *Orthogonal Matrices and Perspective Projections—Revisited*

Let $d \geq 1$ be the distance from the eye point E to the perspective plane S, and choose θ so that $|\csc(\theta)| = d$. Introduce a coordinate system for which S is the plane $z = -\cot(\theta)$, parallel to the

xy-plane with unit normal $N=(0,0,-1)$, and place the eye point E along the z-axis at the location $E=(0,0,\csc(\theta)-\cot(\theta))$, a distance $d=|\csc(\theta)|$ from the plane S. To simplify our notation, let $c=\cos(\theta)$ and $s=\sin(\theta)$. Then by Equation (6.5) since S passes through the point $Q=(0,0,-c/s)$, for any point $P=(x,y,z)$, the line EP intersects the plane S at the point

$$P^{new}=\frac{\big((P-E)\cdot N\big)E+\big((Q-E)\cdot N\big)(P-E)}{(P-E)\cdot N}=\left(\frac{x}{1-c-sz},\ \frac{y}{1-c-sz},\ -\frac{c}{s}\right).$$

By Equation (6.8) the standard singular 4×4 matrix $Persp(E,Q,N)$ representing this perspective projection is given by

$$Persp(E,Q,N)=\begin{pmatrix}\big((Q-E)\cdot N\big)I+N^T*E & N^T\\ -(Q\cdot N)E & -E\cdot N\end{pmatrix}=\begin{pmatrix}\frac{1}{s} & 0 & 0 & 0\\ 0 & \frac{1}{s} & 0 & 0\\ 0 & 0 & \frac{c}{s} & -1\\ 0 & 0 & \frac{c(1-c)}{s^2} & \frac{c-1}{s}\end{pmatrix},$$

and it is straightforward to verify that

$$\begin{aligned}(P-E,0)*Persp(E,Q,N)&=(x,y,z-\frac{1-c}{s},0)*\begin{pmatrix}\frac{1}{s} & 0 & 0 & 0\\ 0 & \frac{1}{s} & 0 & 0\\ 0 & 0 & \frac{c}{s} & -1\\ 0 & 0 & \frac{c(1-c)}{s^2} & \frac{c-1}{s}\end{pmatrix}\\ &=\left(\frac{x}{s},\frac{y}{s},\frac{c(sz+c-1)}{s^2},\frac{-sz-c+1}{s}\right)\\ &\equiv\left(\frac{x}{1-c-sz},\ \frac{y}{1-c-sz},\ -\frac{c}{s}\right).\end{aligned}$$

Since the map $T_{q(N,-\theta/2)}$ represents a rotation in four dimensions, it follows by Theorem 6.9 that there is a 4×4 orthogonal matrix R corresponding to the sandwiching map $T_{q(N,-\theta/2)}$ such that

$$(P-E,0)*R=(P,1)*Persp(E,Q,N).$$

The map $T_{q(N,-\theta/2)}$ represents a rotation in four dimensions in the O, N plane. Since the plane of projection S is parallel to the xy-plane and since the unit normal N points along the negative z-axis, the plane of O, N in four dimensions is the $w(-z)$-plane. Thus in four dimensions the sandwiching map $T_{q(N,-\theta/2)}$ is just rotation by $-\theta$ in the $w(-z)$-plane or equivalently rotation by $-\theta$ in the zw-plane. Therefore the 4×4 orthogonal matrix R corresponding to the map $T_{q(N,-\theta/2)}$ is given by

$$R=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix}$$

To check that R correctly performs perspective projection, observe that

$$\begin{aligned}(P-E,0)*R &= \left(x, y, \frac{sz+c-1}{s}, 0\right)*\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix} \\ &= \left(x, y, \frac{c(sz+c-1)}{s}, -(sz+c-1)\right) \\ &\equiv \left(\frac{x}{1-c-sz}, \frac{y}{1-c-sz}, -\frac{c}{s}\right)\end{aligned}$$

as required. For further details on the relationship between perspective projections and 4×4 orthogonal matrices, see Section 8.4.

Theorem 6.9 is stated and proved only for a special configuration of the eye point E and the plane of projection S. We shall call this position of the eye point E and the plane of projection S the canonical position for the angle θ and the unit normal vector N, and we shall denote these canonical positions by

$$\begin{aligned} E(N,\theta) &= O+(\cot(\theta)-\csc(\theta))N \\ S(N,\theta) &= \text{plane with unit normal } N \text{ at a distance } d=\csc(\theta) \\ &\qquad \text{from the point } E(N,\theta) \text{ along the vector } N. \end{aligned}$$

Now the rather astonishing observation is that a simple variant of Theorem 6.9 remains valid even if the eye point E and the plane of projection S are in arbitrary rather than canonical position.

Projecting a scene from an eye point E into a plane S and then translating the projection by a vector v into the plane $S+v$ is equivalent to first translating the entire scene by the vector v and

then projecting from the translated eye point $E + v$ into the translated plane $S + v$. Indeed if we translate the entire configuration in Figure 6.5 by the vector v, then we can see immediately that translation and projection commute (for an algebraic proof, see Exercise 6.6). In particular, translating the entire configuration in Figure 6.5 by the vector $E(N, \theta) - E$, where $d = \csc(\theta)$ is the distance from the eye point E to the plane of projection S, we arrive at Figure 6.8 (see Figure 6.9). Therefore we have the following generalization of Theorem 6.9.

Theorem 6.10: *Sandwiching vectors to the eye with $q(N, -\theta/2)$ gives perspective.*

Suppose that $0 < \theta < \pi$ and let

- *E = eye point*
- *S = plane of projection*
- *N = unit normal to the plane of projection*
- *$d = \csc(\theta)$ = distance from the eye point E to the plane of projection S along the normal N*
- *P = a point in three dimensions*

Then

- *$T_{q(N,-\theta/2)}(P - E) = q(N, -\theta/2)\,(P - E)\,q(N, -\theta/2)$ is a mass-point, where*
  - *the point is located at the perspective projection of the point P from the eye point E onto the plane S translated by the vector $E(N, \theta) - E$ to the canonical plane $S(N, \theta)$;*
  - *the mass is equal to $d\sin(\theta)$, where d is the distance of the point P from the plane through the eye point E perpendicular to the unit normal N.*

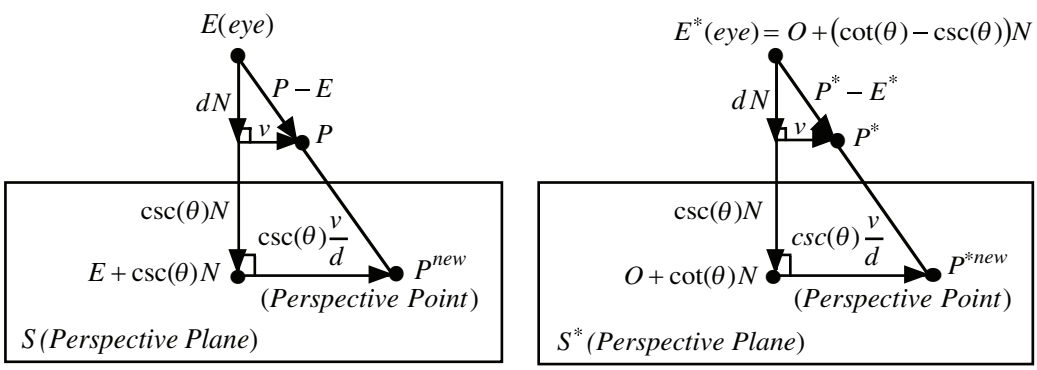


FIGURE 6.9: Translation and perspective commute, since by translation the figure on the left can be made to coincide with the figure on the right.

Proof: Projecting a point P from the eye point E into the plane S and then translating to the canonical plane $S(N, \theta)$ by the vector $e = E(N, \theta) - E$ is equivalent to first translating the point by the vector $e = E(N, \theta) - E$ and then projecting from the canonical eye point $E(N, \theta)$ into the canonical plane $S(N, \theta)$. Now

$$\begin{aligned} q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2) &= q(N,-\theta/2)\,((P+e)-(E+e))\,q(N,-\theta/2) \\ &= q(N,-\theta/2)\,((P+e-E(N,\theta))\,q(N,-\theta/2), \end{aligned}$$

and by Theorem 6.9 the right-hand side represents translating P by the vector e and then projecting from the canonical eye point $E(N, \theta)$ into the canonical plane $S(N, \theta)$. Therefore, since projecting into the plane S and then translating to the canonical plane is equivalent to first translating to canonical position and then projecting into the canonical plane, the left-hand side represents projecting from the eye point E into the plane S and then translating the result by the vector $e = E(N, \theta) - E$ to the canonical plane $S(N, \theta)$. ♦

By Theorem 6.10, if we display the points generated by the sandwiching transformation

$$T_{q(N,-\theta/2)}\,(P-E) = q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2),$$

the scene will appear in perspective. But the scene materializes in the canonical plane rather than in the specified plane of projection. Thus for fixed values of the unit normal vector N and the scalar distance $d = \csc(\theta)$, the scene always appears in the identical plane independent of the absolute position of the eye point E and the plane of projection S. Only the relative positions of E and S matter, not their absolute locations relative to a fixed coordinate system. Typically we are interested only in viewing the scene in perspective; translating the plane of perspective does not matter. Thus, rather remarkably, we can hard code the location of the viewing plane $S(N, \theta)$ depending only on the values of N and θ and not worry about the absolute locations of the eye point E and the plane of projection S.

Exercises:

6.4. Let

- S = the plane through the point $O + \cot(\theta)N$ perpendicular to the unit normal N
- $E = O + (\cot(\theta) - \csc(\theta))N$ = eye point
- P = a point in three dimensions

Show that

$$T_{q(N,-\theta/2)}\,(P-E) = \sin(\theta)((P-Q)\cdot N)\,E + (Q-E)\cdot N)\,P,\,(P-E)\cdot N).$$

6.5. Consider the canonical eye point and the canonical plane for N, θ given by

$$E = E(N, \theta) = O + (\cot(\theta) - \csc(\theta))N$$

$$S = S(N, \theta) = \text{plane with unit normal } N \text{ at a distance } d = \csc(\theta) \text{ from the point } E(N, \theta) \text{ along the vector } N.$$

a. Show that
$Q = Q(N, \theta) = O + \cot(\theta)N$
is the orthogonal projection of the eye point $E(N, \theta)$ onto the plane $S(N, \theta)$.

b. We call the line segment EQ the *axis* of the eye point $E(N, \theta)$ and the plane $S(N, \theta)$. Show that for $0 < \theta \leq \pi/2$:

i. the origin $O = (0, 0, 0)$ lies on the axis of the eye point $E(N, \theta)$ and the plane $S(N, \theta)$.

ii. $\dfrac{|OQ|}{|EQ|} = \cos(\theta)$

iii. O splits the axis into two segments of ratio $\dfrac{1 - \cos(\theta)}{\cos(\theta)}$.

c. How do the results in part b change when $\pi/2 < \theta < \pi$?

6.6. Recall that the formula for projecting a point P from an eye point E into the plane S passing through a point Q and perpendicular to a unit vector N is given by Equation (6.6):

$$P^{new} = \frac{\big((P - Q) \cdot N\big)E + \big((Q - E) \cdot N\big)P}{(P - E) \cdot N}.$$

a. Show that

$$P^{new} + v = \frac{\big((P - Q) \cdot N\big)(E + v) + \big((Q - E) \cdot N\big)(P + v)}{(P - E) \cdot N}.$$

b. Conclude from part a that projecting a point P from the eye point E into the plane S and then translating the resulting point by the vector v is equivalent to first translating the point P by the vector v and then projecting the resulting point from the translated eye point $E + v$ into the translated plane $S + v$. Thus conclude that projection and translation commute.

6.4 ROTORPERSPECTIVES AND ROTOREFLECTIONS

We have studied in detail the two sandwiching maps S_q and T_q. The maps S_q—sandwiching a vector between a unit quaternion q and its conjugate q^*—rotates vectors in three dimensions; the maps T_q—sandwiching a vector between two copies of the same unit quaternion q—computes either

mirror images or perspective projections. But what is the geometric meaning in three dimensions of simple left or simple right multiplication?

Let

$$q = q(N, \theta) = \cos(\theta) O + \sin(\theta) N$$

and consider the two maps:

$$L_q(p) = qp \qquad \text{left multiplication by } q$$
$$R_q(p) = pq \qquad \text{right multiplication by } q.$$

The maps L_q and R_q represent double isoclinic rotations, transformations that rotate quaternions in four dimensions both in the plane of O, N and in the plane perpendicular to O, N by the same absolute angle θ (see Figure 6.3). Therefore, in three dimensions, we should expect that these maps will project like T_q and rotate like S_q.

To see that both of these transformations—rotation and projection—do indeed occur during left and right multiplication, let $r = q(N, \theta/2)$. The quaternion r is the square root of the quaternion q in the plane O, N, which is isomorphic to the complex plane. Hence

$$r^2 = q$$

$$r r^* = O = r^* r.$$

Therefore

$$q p = r (r p r^*) r = r (r p r) r^*,$$

so

$$L_q(p) = T_r(S_r(p)) = S_r(T_r(p)).$$

Thus on vectors in three dimensions left multiplication is indeed the composite of a rotation and a perspective projection, and these operations can be computed in either order. We call such maps *rotorperspectives*.

There are two interesting special cases: $\theta = \pi/2$ and $\theta = \pi$. If $\theta = \pi/2$, then $q = q(N, \pi/2) = N$, and $r = q(N, \pi/4)$. Thus the product of two vectors

$$-(N \cdot v)\, O + N \times v = Nv = L_N(v) = T_{q(N,\, \pi/4)}(S_{q(n,\, \pi/4)}(v))$$

has the following geometric interpretation in three dimensions: First rotate the vector v by the angle $\pi/2$ around the axis vector N to form the vector

$$S_{q(N,\, \pi/4)}(v) = (v \cdot N)\, N + N \times v.$$

Then project the point $P = O + N + (v \cdot N)N + N \times v$ from the eye point $E = O + N$ onto the plane passing through the point O perpendicular to the unit vector N to generate the mass-point

$$T_{q\,(N,\,\pi/4)}\,((v \cdot N)N + N \times v) = -(N \cdot v)\,O + N \times v = Nv$$

located at

$$P = O - \frac{N \times v}{N \cdot v}$$

with mass

$$m = -N \cdot v$$

(see Theorem 6.9). With this interpretation, the product of two vectors Nv is a rotorperspective.

In the special case where $\theta = \pi$, we have $q = q(\mathrm{N}, \pi) = -O$, so

$$L_q(v) = L_{-o}(v) = -v.$$

In this case, $r = q(N, \pi/2) = N$, so $S_r = S_{q(N,\,\pi/2)}$ is rotation by the angle π around the unit vector N, and $T_r = T_N$ is reflection in the plane perpendicular to N. A rotation followed by a reflection is called a *rotoreflection*, so in the special case when $\theta = \pi$, left multiplication is a rotoreflection instead of a rotorperspective. Analogous results hold for right multiplication.

More generally, we can consider the maps

$$S_{q_1, q_2}(w) = q_1 w q_2$$

where $q_1 = q(N, \theta_1)$ and $q_2 = q(N, \theta_2)$. Let $r_1 = q(N, (\theta_1 + \theta_2)/2)$ and $r_2 = q(N, (\theta_1 - \theta_2)/2)$. Then by Lemma 5.1,

$$r_2 r_1 = q_1$$

$$r_1 r^*_2 = q_2.$$

Therefore

$$q_1 p\, q_2 = r_2 (r_1 p\, r_1) r^*_2$$

so

$$S_{q_1, q_2}(p) = S_{r_2}\left(T_{r_1}(p)\right).$$

Thus the map S_{q_1,q_2} is also a rotorperspective: a composite of a rotation and a perspective projection. One interesting special case is when $\theta_1 + \theta_2 = \pi$. In this case, $r_1 = q(N, \pi/2) = N$, so $T_{r_1} = T_N$ is reflection in the plane perpendicular to N, and $S_{r_2} = S_{q(N,(\theta_1-\theta_2)/2)}$ is rotation by the angle $\theta_1 - \theta_2$ around the unit vector N. Hence the maps $T_{r_1} = T_N$ and $S_{r_2} = S_{q(N,(\theta_1-\theta_2)/2)}$ commute. Thus the map $S_{q_1,q_2}(w)$ is a rotoreflection, since $S_{q_1,q_2}(w)$ rotates the vector w around the axis N by the angle $\theta_1 - \theta_2$ and then reflects the resulting vector in the plane perpendicular to N.

Finally, consider the maps

$$S_{q_1,q_2}(w) = q_1 w q_2$$

where $q_1 = q(N_1, \theta_1)$, $q_2 = q(N_2, \theta_2)$, and $N_1 \neq N_2$. To analyze these maps, notice that

$$q_1 p q_2 = q_1 q_2 (q_2^* p q_2)$$

so

$$S_{q_1,q_2}(p) = L_{q_1 q_2}\left(S_{q_2^*}(p)\right).$$

Thus the map S_{q_1,q_2} is the composite of a rotation and a left multiplication—a rotorperspective, which is the composite of another rotation, but around a different axis, and a perspective projection.

• • • •

CHAPTER 7

Recapitulation: Insights and Results

Before we go on to discuss some computational issues, let us pause here for a moment to catch our breath and to recapitulate some of our main theoretical insights and results.

Multiplication by a unit quaternion rotates vectors in four dimensions because multiplication by a unit quaternion is a linear isometry. The key to understanding the precise geometric effect of multiplication by the unit quaternion

$$q(N, \theta) = \cos(\theta)\, O + \sin(\theta)\, N$$

is to study the rotations in four dimensions induced by $q(N, \theta)$ in the plane of O, N and in the plane orthogonal to O, N.

The plane of O, N is isomorphic to the complex plane C: $O \leftrightarrow 1$ and $N \leftrightarrow i$. Therefore multiplication by the unit quaternion $q(N, \theta)$ rotates quaternions in this plane counterclockwise by the angle θ, and left and right multiplication commute. In contrast, the plane of quaternions orthogonal to O, N is isomorphic to Cv, where v is any unit vector perpendicular to N. Moreover, for any complex number q in the plane of O, N

$$vq = q^* v.$$

Therefore in the plane orthogonal to O, N, multiplication by $q(N, \theta)$ on the left represents rotation by the angle θ, whereas multiplication by $q(N, \theta)$ on the right represents rotation by the angle $-\theta$. Thus left and right multiplication by $q(N, \theta)$ represent double isoclinic rotations in four dimensions (see Figure 5.3).

By using sandwiching to combine these 4-dimensional rotations, we can generate simple rotations in four dimensions, leaving quaternions in one of these two planes fixed while rotating quaternions in the complementary plane. This approach allows us to harness quaternion multiplication to rotate and reflect vectors in three dimensions. Moreover, rather remarkably, we can also apply sandwiching—simple rotations in four dimensions—to compute 3-dimensional perspective projections.

To generate our main results, we applied the following key insights:

1. *Quaternions are mass-points.*
   a. A quaternion is not merely the sum of a scalar and a vector. Rather quaternions have a physical–geometric interpretation compatible with the standard mass-point model of space used in contemporary computer graphics.
2. *Multiplying arbitrary quaternions by a fixed unit quaternion rotates quaternions in four dimensions.*
   a. Multiplication by a unit quaternion is a linear transformation because multiplication distributes through addition.
   b. Multiplication by a unit quaternion is an isometry because the length of the product of two quaternions is equal to the product of their lengths.
3. *The identity O for quaternion multiplication represents a prescribed point, the origin for points in three dimensions.*
4. *The plane of O, N has special properties:*
   a. The plane of O, N is isomorphic to the complex plane. Therefore multiplication by unit quaternions $q(N, \theta)$ rotates quaternions in this plane by the angle θ, and left and right multiplication commute.
   b. The plane of O, N in four dimensions represents a line of points in three dimensions, the line through the point O parallel to the vector N.
   c. Rotations in four dimensions that are the identity on the plane of O, N are the identity on a line in three dimensions. Therefore rotations of quaternions that are the identity on the plane of O, N in four dimensions correspond to rotations about a line in three dimensions.
   d. Simple rotations in the plane of O, N in four dimensions are the identity on the plane of vectors orthogonal to N in three dimensions.
   e. Simple rotations in the plane of O, N in four dimensions correspond to reflections in and perspective projections onto a plane orthogonal to N in three dimensions.
   f. Rotations of quaternions in the plane of O, N convert distance along N into mass at O. Therefore no information is lost, even though these rotations in four dimensions can correspond to perspective projections in three dimensions. This observation allows us to detect hidden surfaces while performing perspective projection.
5. *The plane perpendicular to O, N also has special properties:*
   a. The plane perpendicular to O, N is isomorphic to the complex numbers times v, where v is any vector perpendicular to N. Moreover, $vq = q^* v$ for any complex number q in the plane of O, N.

b. In the plane perpendicular to O, N, multiplication by the unit quaternions $q(N, \theta)$ on the left represents rotation by θ, whereas multiplication by the unit quaternions $q(N, \theta)$ on the right represents rotation by $-\theta$.

c. The plane of quaternions perpendicular to O, N in four dimensions represents the plane of vectors perpendicular to N in three dimensions.

d. Rotations in four dimensions that are the identity on the plane perpendicular to O, N are the identity on a plane of vectors in three dimensions. Therefore rotations of quaternions that are the identity on the plane perpendicular to O, N in four dimensions correspond to reflections in and projections onto a plane perpendicular to N in three dimensions.

6. *Sandwiching is the basic orthogonal transformation in four dimensions.*

a. Sandwiching represents simple rotations in four dimensions, unlike left and right multiplication which represent double isoclinic rotations in four dimensions.

b. Sandwiching measures the anticommutativity of quaternion multiplication. Therefore we can harness sandwiching to fix complementary planes in four dimensions.

i. Sandwiching a quaternion between a unit quaternion $q(N, \theta)$ and its conjugate $q^*(N, \theta)$ is the identity on quaternions in the plane of O, N.

ii. Sandwiching a quaternion between two copies of a unit quaternion $q(N, \theta)$ is the identity on the quaternions in the plane perpendicular to O, N.

• • • •

PART II

Computation

CHAPTER 8

Matrix Representations for Rotations, Reflections, and Perspective Projections

Quaternion multiplication is a linear transformation on the 4-dimensional vector space of mass-points because quaternion multiplication distributes through quaternion addition. Therefore both left and right quaternion multiplication can be represented by 4×4 matrices. Here we shall begin our study of computational issues by deriving the matrices representing left and right quaternion multiplication. We shall then simply multiply these matrices together to derive matrix representations for rotations, reflections, and perspective projections (see also Shoemake, 1991).

8.1 MATRIX REPRESENTATIONS FOR QUATERNION MULTIPLICATION

Consider two quaternions

$$p = p_4 O + p_1 i + p_2 j + p_3 k = (p_1, p_2, p_3, p_4)$$
$$q = q_4 O + q_1 i + q_2 j + q_3 k = (q_1, q_2, q_3, q_4).$$

Recall that by Equation (4.12)

$$\begin{aligned} pq &= (p_4q_4 - p_1q_1 - p_2q_2 - p_3q_3)\,O + (p_4q_1 + p_1q_4)\,i + (p_4q_2 + p_2q_4)\,j + \\ &\quad (p_4q_3 + p_3q_4)\,k + (p_2q_3 - p_3q_2)\,i + (p_3q_1 - p_1q_3)\,j + (p_1q_2 - p_2q_1)\,k \\ &= (p_4q_1 + p_1q_4 + p_2q_3 - p_3q_2, p_4q_2 + p_2q_4 + p_3q_1 - p_1q_3, p_4q_3 + p_3q_4 \\ &\quad + p_1q_2 - p_2q_1, p_4q_4 - p_1q_1 - p_2q_2 - p_3q_3) \end{aligned} \tag{8.1}$$

Define

$$L(q) = \begin{pmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix} \tag{8.2}$$

$$R(q) = \begin{pmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}. \tag{8.3}$$

Then using Equations (8.1)–(8.3), it is straightforward to verify by direct matrix multiplication that

$$qp = p * L(q) \qquad \text{left multiplication by } q. \tag{8.4}$$

$$pq = p * R(q) \qquad \text{right multiplication by } q \tag{8.5}$$

(see Exercise 8.1). Notice that the rows of $L(q)$ and $R(q)$ are orthogonal and their determinants are $|q|^4$. Thus if q is a unit quaternion, then $L(q)$ and $R(q)$ are orthogonal matrices and therefore—as expected since quaternion multiplication preserves length—represent rotations in four dimensions.

We have seen in Chapter 6 that rotation, reflection, and perspective projection in three dimensions are each just composites of left and right quaternion multiplication (with and without conjugation), so we can simply multiply the matrices $L(q)$ and $R(q)$ (or $R(q^*)$) to find explicit matrix representations for these three transformations. Next we shall investigate each of these transformations in turn.

Exercises:

8.1. Let $L(q)$ and $R(q)$ be the matrices defined in Equations (8.2) and (8.3). Verify that

a. $qp = p * L(q)$
b. $pq = p * R(q)$

8.2. Let $L(q)$ and $R(q)$ be the matrices defined in Equations (8.2) and (8.3). Show that

a. $L(p+q) = L(p) + L(q)$
b. $R(p+q) = R(p) + R(q)$
c. $L(pq) = L(q)\,L(p)$
d. $R(pq) = R(p)\,R(q)$
e. $L(p)R(q) = R(q)L(p)$.

(Hint: Interpret each of these results in terms of quaternion multiplication.)

8.3. Let $L(q)$ and $R(q)$ be the matrices defined in Equations (8.2) and (8.3), and let I be the 4×4 identity matrix. By direct computation verify that if q is a unit quaternion, then

a. $L(q) * L(q)^T = I = R(q) * R(q)^T$
b. $\det(L(q)) = 1 = \det(R(q))$.

8.2 MATRIX REPRESENTATIONS FOR ROTATIONS

Recall from Theorem 6.1 that the map

$$S_{q(N,\,\theta/2)}(v) = q(N,\,\theta/2)\; v\; q^*(N,\,\theta/2) \tag{8.6}$$

rotates the vector v by the angle θ around the axis N. Let

$$q = q\,(N,\,\theta/2) = (q_1,\,q_2,\,q_3,\,q_4).$$

Then by Equations (8.4)–(8.6)

$$S_q(v) = v * L(q) * R(q^*).$$

Therefore the matrix $Rot(N,\,\theta)$ representing rotation around the axis vector N by the angle θ is given by

$$\begin{aligned} Rot(N,\theta) &= L(q) * R(q^*) \\ &= \begin{pmatrix} q_4^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2+2q_3q_4 & 2q_1q_3-2q_2q_4 & 0 \\ 2q_1q_2-2q_3q_4 & q_4^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3+2q_1q_4 & 0 \\ 2q_1q_3+2q_2q_4 & 2q_2q_3-2q_1q_4 & q_4^2-q_1^2-q_2^2+q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{8.7}$$

Thus it is easy to extract the rotation matrix $Rot(N,\,\theta)$ from the components of the unit quaternion $q(N,\,\theta/2)$. Notice too that since

$$q(N,\,\theta/2) = \cos(\theta/2)O + \sin(\theta/2)N = (\sin(\theta/2)N_1,\,\sin(\theta/2)N_2,\,\sin(\theta/2)N_3,\,\cos(\theta/2)),$$

it follows from substitution into Equation (8.7) and the half angle formulas that

$$Rot(N,\theta) = \begin{pmatrix} \cos(\theta)+(1-\cos(\theta))N_1^2 & (1-\cos(\theta))N_1N_2+\sin(\theta)N_3 & (1-\cos(\theta))N_1N_3-\sin(\theta)N_2 & 0 \\ (1-\cos(\theta))N_1N_2-\sin(\theta)N_3 & \cos(\theta)+(1-\cos(\theta))N_2^2 & (1-\cos(\theta))N_2N_3+\sin(\theta)N_1 & 0 \\ (1-\cos(\theta))N_1N_3+\sin(\theta)N_2 & (1-\cos(\theta))N_2N_3-\sin(\theta)N_1 & \cos(\theta)+(1-\cos(\theta))N_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.8}$$

Conversely, given the rotation matrix $R = Rot(N,\,\theta)$, we can find the corresponding unit quaternion $q = q(N,\,\theta/2)$ in the following fashion. From Equation (8.7)

$$3q_4^2 - q_1^2 - q_2^2 - q_3^2 = R_{11} + R_{22} + R_{33}.$$

Therefore since

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = |q|^2 = 1,$$

it follows that

$$4q_4^2 - 1 = R_{11} + R_{22} + R_{33}.$$

Hence

$$q_4 = \frac{\sqrt{R_{11} + R_{22} + R_{33} + 1}}{2}. \tag{8.9}$$

Moreover again by Equation (8.7)

$$\begin{aligned} 4\,q_1 q_4 &= R_{23} - R_{32} \\ 4\,q_2 q_4 &= R_{31} - R_{13} \\ 4\,q_3 q_4 &= R_{12} - R_{21}. \end{aligned} \tag{8.10}$$

so by Equations (8.9) and (8.10)

$$\begin{aligned} q_1 &= \frac{R_{23} - R_{32}}{2\sqrt{R_{11} + R_{22} + R_{33} + 1}} \\ q_2 &= \frac{R_{31} - R_{13}}{2\sqrt{R_{11} + R_{22} + R_{33} + 1}} \\ q_3 &= \frac{R_{12} - R_{21}}{2\sqrt{R_{11} + R_{22} + R_{33} + 1}}. \end{aligned} \tag{8.11}$$

Equations (8.9) and (8.11) are valid provided that $q_4 = \cos(\theta/2) \neq 0$—that is, provided that $\theta \neq \pi$. If $\theta \approx \pi$, then as in Equation (8.9) we can compute

$$\begin{aligned} q_1 &= \frac{\sqrt{R_{11} - R_{22} - R_{33} + 1}}{2} \\ q_2 &= \frac{\sqrt{-R_{11} + R_{22} - R_{33} + 1}}{2} \\ q_3 &= \frac{\sqrt{-R_{11} - R_{22} + R_{33} + 1}}{2}. \end{aligned} \tag{8.12}$$

For numerical stability choose the largest of these three values, say q_1. Then as in Equation (8.11)

$$q_2 = \frac{R_{12} + R_{21}}{2\sqrt{R_{11} - R_{22} - R_{33} + 1}}$$
$$q_3 = \frac{R_{13} + R_{31}}{2\sqrt{R_{11} - R_{22} - R_{33} + 1}}$$
$$q_4 = \frac{R_{23} - R_{32}}{2\sqrt{R_{11} - R_{22} - R_{33} + 1}}. \tag{8.13}$$

Exercises:

8.4. By direct computation, verify that if $q = q_4O + q_1i + q_2j + q_3k$ is a unit quaternion, then

$$L(q) * R(q^*) = \begin{pmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_3q_4 & 2q_1q_3 - 2q_2q_4 & 0 \\ 2q_1q_2 - 2q_3q_4 & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_1q_4 & 0 \\ 2q_1q_3 + 2q_2q_4 & 2q_2q_3 - 2q_1q_4 & q_4^2 - q_1^2 - q_2^2 + q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8.5. Let $q = q_4O + q_1i + q_2j + q_3k$.

a. By direct computation show that

i. $q\,i\,q^* = \left(q_4^2 + q_1^2 - q_2^2 - q_3^2, 2q_1q_2 + 2q_3q_4, 2q_1q_3 - 2q_2q_4\right)$

ii. $q\,j\,q^* = \left(2q_1q_2 - 2q_3q_4, q_4^2 - q_1^2 + q_2^2 - q_3^2, 2q_2q_3 + 2q_1q_4\right)$

iii. $q\,k\,q^* = \left(2q_1q_3 + 2q_2q_4, 2q_2q_3 - 2q_1q_4, q_4^2 - q_1^2 - q_2^2 + q_3^2\right)$

iv. $q\,O\,q* = O$.

b. Conclude from part *a* that if $q = q(N, \theta/2)$, then

$$Rot(N,\theta) = \begin{pmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_3q_4 & 2q_1q_3 - 2q_2q_4 & 0 \\ 2q_1q_2 - 2q_3q_4 & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_1q_4 & 0 \\ 2q_1q_3 + 2q_2q_4 & 2q_2q_3 - 2q_1q_4 & q_4^2 - q_1^2 - q_2^2 + q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8.6. Let N^T denote the column vector that is the transpose of the row vector N, and let I denote the 3×3 identity matrix.

a. Using Equation (8.8), show that

$$Rot(N,\theta) = \cos(\theta)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \left(1 - \cos(\theta)\right)(N^T * N) & 0 \\ 0 & 1 \end{pmatrix} + \sin(\theta)\begin{pmatrix} 0 & N_3 & -N_2 & 0 \\ -N_3 & 0 & N_1 & 0 \\ N_2 & -N_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b. Using part a, prove the following formula of Rodrigues:
$(v,0) * Rot(N, \theta) = \cos(\theta)v + (1 - \cos(\theta))(N \cdot v)N + \sin(\theta)N \times v.$

c. Using part b, show that
$(N, 0) * Rot(N, \theta) = (N, 0).$

8.3 MATRIX REPRESENTATIONS FOR MIRROR IMAGES

Recall from Theorem 6.4 that the map

$$T_N(v) = N\,v\,N \tag{8.14}$$

is the mirror image of v in the plane perpendicular to N. Hence by Equations (8.4), (8.5)

$$T_N(v) = v * L(\mathrm{N}) * R(\mathrm{N}).$$

Therefore the matrix $Mir(N)$ representing mirror image in the plane perpendicular to the unit vector N is given by

$$\begin{aligned} Mir(N) &= L(N) * R(N) \\ &= \begin{pmatrix} N_2^2 + N_3^2 - N_1^2 & -2N_1N_2 & -2N_1N_3 & 0 \\ -2N_1N_2 & N_1^2 - N_2^2 + N_3^2 & -2N_2N_3 & 0 \\ -2N_1N_3 & -2N_2N_3 & N_1^2 + N_2^2 - N_3^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{aligned} \tag{8.15}$$

or equivalently since N is a unit vector

$$Mir(N) = \begin{pmatrix} 1 - 2N_1^2 & -2N_1N_2 & -2N_1N_3 & 0 \\ -2N_1N_2 & 1 - 2N_2^2 & -2N_2N_3 & 0 \\ -2N_1N_3 & -2N_2N_3 & 1 - 2N_3^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8.16}$$

Conversely, given the mirror image matrix $M = Mir(N)$, we can easily find the corresponding unit normal vector N since by Equation (8.16)

$$N_k = \sqrt{\frac{1 - M_{kk}}{2}},\ k = 1,2,3. \tag{8.17}$$

Notice the value -1 in the lower right-hand corner of the matrix $M = Mir(N)$. If $v = v_1 i + v_2 j + v_3 k = (v_1, v_2, v_3, 0)$ is a vector in $\boldsymbol{R}^3$, then the upper 3×3 submatrix of M computes the mirror

image of v in the plane perpendicular to N; the fourth component of v is zero, so the value -1 in the lower right-hand corner of M does not affect the value of $v * M$. On the other hand, for points $P = O + p_1 i + p_2 j + p_3 k = (p_1, p_2, p_3, 1)$ in $\boldsymbol{R}^3$, the value -1 in the lower right-hand corner of M does affect the value of $P * M$, and $P * M$ is not the mirror image of the point P in the plane through O perpendicular to the vector N. Indeed as we observed in Section 6.2

$$T_N(P) = NPN$$

is not the mirror image of P; rather the mirror image of P is given by

$$T_N(P - 2O) = N(P - 2O)N = (P - 2O) * M.$$

But $P - 2O = (p_1, p_2, p_3, -1)$, so we need to change the fourth component of points P from $+1$ to -1 to correctly compute the mirror image of P using the matrix M. Alternatively, we can simply change the sign of M_{44} instead of the sign of p_4 and get the same result. The original value $M_{44} = -1$ because $M = L(N) * R(N)$ represents a rotation in $\boldsymbol{R}^4$, so $\det(M) = 1$. Since the upper 3×3 submatrix of M has determinant -1, to balance this sign $M_{44} = -1$. But if we simply dispense with this minus sign, then we can use the same matrix to correctly compute the mirror image of both points and vectors in $\boldsymbol{R}^3$.

Exercises:

8.7. By direct computation, verify that if $N = (N_1, N_2, N_3, 0)$ is a unit vector, then

$$L(N) * R(N) = \begin{pmatrix} 1 - 2N_1^2 & -2N_1N_2 & -2N_1N_3 & 0 \\ -2N_1N_2 & 1 - 2N_2^2 & -2N_2N_3 & 0 \\ -2N_1N_3 & -2N_2N_3 & 1 - 2N_3^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

8.8. Let N^T denote the column vector that is the transpose of the row vector N, and let I denote the 3×3 identity matrix.

a. Show that

$$Mir(N) = \begin{pmatrix} I - 2N^T * N & 0 \\ 0 & -1 \end{pmatrix}.$$

b. Using part a, show that

i. $v \perp N \Rightarrow (v, 0) * Mir(N) = (v, 0)$

ii. $(N, 0) * Mir(N) = (-N, 0)$.

8.4 MATRIX REPRESENTATIONS FOR PERSPECTIVE PROJECTIONS

Let

- E = eye point
- S = plane of projection
- N = unit normal to the plane of projection
- $d = \csc(\theta)$ = distance from the eye point E to the plane of projection S along the normal N
- P = a point in three dimensions

Then by Theorem 6.10

- $T_{q(N,-\theta/2)}(P-E) = q(N,-\theta/2)\,(P-E)\,q(N,-\theta/2)$ is a mass-point, where
  —the point is located at the perspective projection of the point P from the eye point E onto the plane S translated by the vector $E(N,\theta) - E$ to the canonical plane $S(N,\theta)$
  —the mass is equal to $d\sin(\theta)$, where d is the distance of the point P from the plane through the eye point E perpendicular to the unit normal N.

Now let $v = P - E$ and $q = q(N, -\theta/2)$. Then by Equations (8.4) and (8.5)

$$T_q(v) = v * L(q) * R(q).$$

Therefore the matrix $Persp(N,\theta)$ representing perspective projection from the eye point E into the plane S translated to the canonical plane $S(N,\theta)$ is given by the matrix

$$Persp(N,\theta) = L(q) * R(q) = \begin{pmatrix} 1-2q_1^2 & -2q_1q_2 & -2q_1q_3 & -2q_1q_4 \\ -2q_1q_2 & 1-2q_2^2 & -2q_2q_3 & -2q_2q_4 \\ -2q_1q_3 & -2q_2q_3 & 1-2q_3^2 & -2q_3q_4 \\ 2q_1q_4 & 2q_2q_4 & 2q_3q_4 & 2q_4^2-1 \end{pmatrix} \tag{8.18}$$

Notice too that since

$$\begin{aligned} q(N,-\theta/2) &= \cos(\theta/2)O - \sin(\theta/2)N \\ &= (-\sin(\theta/2)N_1, -\sin(\theta/2)N_2, -\sin(\theta/2)N_3, \cos(\theta/2)) \end{aligned}$$

it follows from substitution into Equation (8.18) and the half angle formulas that

$$Persp(N,\theta)=\begin{pmatrix} 1+(\cos(\theta)-1)N_1^2 & (\cos(\theta)-1)N_1N_2 & (\cos(\theta)-1)N_1N_3 & \sin(\theta)N_1 \\ (\cos(\theta)-1)N_1N_2 & 1+(\cos(\theta)-1)N_2^2 & (\cos(\theta)-1)N_2N_3 & \sin(\theta)N_2 \\ (\cos(\theta)-1)N_1N_3 & (\cos(\theta)-1)N_2N_3 & 1+(\cos(\theta)-1)N_3^2 & \sin(\theta)N_3 \\ -\sin(\theta)N_1 & -\sin(\theta)N_2 & -\sin(\theta)N_3 & \cos(\theta) \end{pmatrix}. \tag{8.19}$$

For example, when $N = (0, 0, -1)$,

$$Persp(N,\theta)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

which coincides with the result in Example 6.2.

Rather than express the matrix for perspective projection in terms of trigonometric functions of the quaternion angle θ, geometrically it is more natural to write this matrix as a function of the distance d between the eye point E and the plane of projection S. We can easily convert between the distance d and the angle θ because by construction

$$d = \csc(\theta).$$

Moreover, if $\csc(\theta) = d$, then $\sin(\theta) = 1/d$ and $\cos(\theta) = \pm\sqrt{1-1/d^2}$. Substituting these expressions for $\cos(\theta)$, $\sin(\theta)$ in terms of d into Equation (8.19), we find that in terms of the distance d and the unit normal vector N the matrix for perspective projection is given by

$$Persp(N,d)=\begin{pmatrix} 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_1^2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_3 & N_1/d \\ \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_2 & 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_2^2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_2N_3 & N_2/d \\ \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_3 & \left(\pm\sqrt{1-1/d^2}-1\right)N_2N_3 & 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_3^2 & N_3/d \\ -N_1/d & -N_2/d & -N_3/d & \pm\sqrt{1-1/d^2} \end{pmatrix} \tag{8.20}$$

For example, if $d = 1$ and $N = (0, 0, -1)$ then

$$Persp(N,\ d)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which coincides with the result in Example 6.1.

Conversely, given the perspective matrix $P = Persp(N, \theta)$, we can find the corresponding quaternion $q = q(\mathrm{N}, -\theta/2)$ in the following fashion. From Equation (8.18)

$$q_4 = \sqrt{\frac{P_{44}+1}{2}}. \tag{8.21}$$

Moreover, by assumption, $q_{44} \neq 0$, since otherwise q would represent mirror image, not perspective projection. Now by Equation (8.18)

$$q_k = \frac{P_{4k} - P_{k4}}{4q_4}, \; k = 1, 2, 3 \tag{8.22}$$

Finally, notice that to compute perspective projection, we typically apply the matrix $Persp(N, \theta)$ only to vectors $v = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, 0)$, rather than to points $P = (p_1, p_2, p_3, 1)$. Therefore only the upper 3×4 entries of $Persp(N, \theta)$ are used to compute perspective projection. The fourth row has no affect on vectors; this row is there only to insure that $Persp(N, \theta)$ is a rotation matrix on $\boldsymbol{R}^4$.

Exercises:

8.9. By direct computation, verify that if $q = q_4 O + q_1 i + q_2 j + q_3 k$ is a unit quaternion, then

$$L(q) * R(q) = \begin{pmatrix} 1-2q_1^2 & -2q_1q_2 & -2q_1q_3 & -2q_1q_4 \\ -2q_1q_2 & 1-2q_2^2 & -2q_2q_3 & -2q_2q_4 \\ -2q_1q_3 & -2q_2q_3 & 1-2q_3^2 & -2q_3q_4 \\ 2q_1q_4 & 2q_2q_4 & 2q_3q_4 & 2q_4^2-1 \end{pmatrix}.$$

8.10. Let N^T denote the column vector that is the transpose of the row vector N, and let I denote the 3×3 identity matrix.

a. Using Equation (8.19), show that

$$Persp(N,\theta) = \begin{pmatrix} I + (\cos(\theta)-1)(N^T * N) & \sin(\theta)N^T \\ -\sin(\theta)N & \cos(\theta) \end{pmatrix}$$

b. Using part a, show that

i. $O * Persp(N, \theta) = -\sin(\theta)N + \cos(\theta)O$

ii. $(v, 0) * Persp(N, \theta) = (v, 0) + (\cos(\theta) - 1)((v \cdot N)N, \sin(\theta)(v \cdot N))$.

c. Conclude from part b that

i. $v \perp N \Rightarrow (v, 0) * Persp(N, \theta) = (v, 0)$

ii. $(N, 0) * Persp(N, \theta) = \cos(\theta)N + \sin(\theta)O$

d. Let $u(N, \phi) = \cos(\phi)N + \sin(\phi)O$. Using parts b and c, show that

i. $u(N, \phi) * Persp(N, \theta) = u(N, \phi + \theta)$.

e. Conclude from parts c, d that the matrix $Persp(N, \theta)$ represents a simple rotation in four dimensions by the angle θ in the plane determined by N, O and is the identity on vectors in the plane perpendicular to N, O.

8.11. Let N^T denote the column vector that is the transpose of the row vector N, and let I denote the 3×3 identity matrix.

a. Using Equation (8.20), show that

$$Persp(N,d) = \begin{pmatrix} I + \left(\pm\sqrt{1-1/d^2} - 1\right)(N^T * N) & N^T/d \\ -N/d & \pm\sqrt{1-1/d^2} \end{pmatrix}.$$

b. Using part a, show that

$$(v,0) * Persp(N,d) = (v,0) + \left(\pm\sqrt{1-1/d^2} - 1\right)\left((v \cdot N)N,\ v \cdot N/d\right)$$

c. Conclude from part b that

i. $v \perp N \Rightarrow (v, 0) * Persp(N, d) = (v, 0)$

ii. $(N,0) * Persp(N, d) = \left(\pm\sqrt{1-1/d^2}\, N, 1/d\right)$.

8.12. Recall from Section 6.3 that the canonical eye point and the canonical plane for N, θ are given by

$E(N, \theta) = O + (\cot(\theta) - \csc(\theta))N$

$S(N, \theta)$ = plane with unit normal N at a distance $d = \csc(\theta)$ from the point $E(N, \theta)$ along the vector N.

a. Show that

$$Lim_{\theta \to \pi}(\cot(\theta) - \csc(\theta)) = \pm\infty.$$

b. Conclude from part a that as $\theta \to \pi$, the eye point $E(N, \theta)$ recedes to infinity in the direction $\pm N$.

c. Using Exercise 8.10, show that

$$Lim_{\theta \to \pi} Persp(N,\theta) = \begin{pmatrix} I - 2N^T * N & 0 \\ 0 & -1 \end{pmatrix},$$

d. Conclude from part c and Exercise 8.8 that

$$Lim_{\theta \to \pi} Persp(N, \theta) = Mir(N).$$

8.13. Recall from Section 6.3 that the canonical eye point and the canonical plane for N, θ are given by

$E = E(N, \theta) = O + (\cot(\theta) - \csc(\theta))N$

$S = S(N, \theta) =$ plane with unit normal N at a distance $d = \csc(\theta)$ from the point $E(N, \theta)$ along the vector N—that is, the plane with unit normal N passing through the point $Q = Q(N, \theta) = O + \cot(\theta)N$

and that

$$Persp(E,Q,N) = \begin{pmatrix} ((Q-E)\cdot N)I + N^T * E & N^T \\ -(Q\cdot N)E & -E\cdot N \end{pmatrix}.$$

Let $M_{3\times 4}$ denote the upper 3×4 submatrix of a 4×4 matrix M. Show that

$$Persp(N, \theta)_{3\times 4} = \sin(\theta)\, Persp(E, Q, N)_{3\times 4}.$$

• • • •

CHAPTER 9

Applications

In this chapter we examine the three main applications of quaternions in computer graphics: to increase speed and reduce storage for calculations involving rotations, reflections, and perspective projections; to avoid distortions arising from numerical inaccuracies caused by floating point computations with rotations; and to interpolate between two rotations for key frame animation.

9.1 EFFICIENCY: QUATERNIONS VERSUS MATRICES

We now have two ways to represent certain important transformations on vectors in three dimensions: matrices and quaternions. For rotations and reflections, we can use 3×3 matrices; for perspective projections we require 3×4 matrices. Here we shall compare and contrast some of the advantages and disadvantages of quaternion and matrix representations for these key transformations. For the purposes of this discussion, we shall assume that we can represent our transformations by 3×3 matrices; the discussion changes only slightly for 3×4 matrices.

A 3×3 matrix contains nine scalar entries, whereas a quaternion can be represented by four rectangular coordinates. Thus if memory is at a premium, quaternions are more than twice as efficient as matrices.

On the other hand, by Equation (4.8) to compute the product of two quaternions requires 16 real scalar multiplications: six for the cross product, three for the dot product, six for the two scalar products, and one for the product of the two scalar masses. Similarly, since a vector has zero mass, the product of a quaternion and a vector requires 12 scalar multiplications because the product of the masses as well as one of the scalar products is zero. Sandwiching a vector between two quaternions computes a product of a quaternion with a vector followed by the product of a quaternion with a quaternion, so the total cost for one transformation using quaternions is $12 + 16 = 28$ scalar multiplications. In contrast, the cost for multiplying a vector by a 3×3 matrix is only nine scalar multiplications. Thus matrices are more than three times as fast as quaternions for computing these transformations.

But there is an additional computational consideration. To compose two rotations by quaternion multiplication requires 16 scalar multiplications, whereas to compose two rotations by matrix multiplication requires 27 scalar multiplications. Therefore quaternion multiplication is more than

TABLE 9.1: Tradeoffs between speed and memory for quaternions and 3 × 3 matrices.

	MEMORY	TRANSFORMATION	COMPOSITION
Quaternions	4 scalars	28 multiplies	16 multiplies
3 × 3 Matrices	9 scalars	9 multiplies	27 multiplies

1.5 times faster than matrix multiplication for composing these transformations. Thus some computations favor quaternions, whereas others favor matrices. We summarize these tradeoffs for both memory and speed in Table 9.1.

9.2 AVOIDING DISTORTION BY RENORMALIZATION

Aside from considerations of memory and speed, one of the main advantages of quaternions over matrices is avoiding distortions that arise from numerical inaccuracies that are inevitably introduced by floating point computations.

For any 3 × 3 matrix A representing an affine transformation,

$$\begin{pmatrix} i_{new} \\ j_{new} \\ k_{new} \end{pmatrix} = \begin{pmatrix} i \\ j \\ k \end{pmatrix} * A \quad .$$

Since the matrix with rows i, j, k is the identity matrix, we conclude that

$$A = \begin{pmatrix} i_{new} \\ j_{new} \\ k_{new} \end{pmatrix}.$$

Hence the rows of A represent the images of the unit vectors i, j, k along the coordinate axes.

Thus if a 3 × 3 matrix A represents a rotation or reflection, then the rows of A must be mutually orthogonal vectors, since these transformations preserve angles and the original vectors i, j, k are mutually orthogonal. To compose these transformations, we must multiply matrices. But the entries of these matrices—especially rotation matrices that contain values of sines and cosines—are typically represented by floating point numbers. Therefore after composing several rotations, the rows of the product matrix will no longer be mutually orthogonal vectors due to numerical inaccuracies

introduced by floating point computations. Thus these matrices no longer represent rotations, so applying these matrices to vectors will distort angles in the image.

Quaternions avoid this problem. To compose rotations, we multiply the corresponding quaternions. Since every quaternion represents a conformal transformation, computing transformations by sandwiching with quaternions will never distort angles.

Suppose, however, that we want to compose only rotations. Composing rotations by matrix multiplication, we will generate distortions in both lengths and angles due to numerical inaccuracies introduced by floating point computations. We need to renormalize the resulting matrix so that the rows are mutually orthogonal unit vectors, but it is not so clear how to perform this renormalization.

This normalization, however, is easy to perform with quaternions. Rotations are represented by unit quaternions. When we compose rotations by multiplying unit quaternions, the result may no longer be a unit quaternion due to floating point computations. But we can normalize the resulting quaternion q simply by dividing by its length $|q|$. The resulting quaternion $q/|q|$ is certainly a unit quaternion, so this renormalization avoids distorting either lengths or angles in the image.

9.3 KEY FRAME ANIMATION AND SPHERICAL LINEAR INTERPOLATION

Quaternions have one additional advantage over matrices: quaternions can be used to interpolate in between frames for key frame animation. In key frame animation, an artist draws only a few key frames in the scene, and an animator must interpolate intermediate frames to make the animation appear natural.

Consider an object tumbling through a scene. An artist will typically draw only a few frames representing rotations of the object at certain key times. An animator must then find intermediate rotations, so that the tumbling looks smooth and natural. Suppose that from the artist's drawing we know the rotations R_0 and R_1 at times $t = 0$ and $t = 1$. How do we find the appropriate intermediate rotations at intermediate times?

If the rotations R_0, R_1 are represented by 3×3 matrices, then we seek intermediate rotation matrices $R(t)$ for $0 < t < 1$. We might try linear interpolation and set

$$R(t) = (1 - t)R_0 + tR_1.$$

The problem with this approach is that the matrices $R(t)$ are no longer rotation matrices because the rows of these matrices are no longer mutually orthogonal unit vectors. Thus the matrices $R(t)$ will introduce undesirable distortions into the animation.

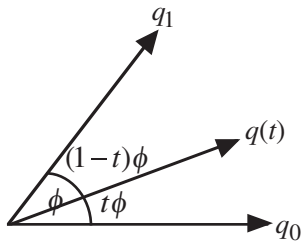


FIGURE 9.1: Spherical linear interpolation (SLERP): The quaternion $q(t)$ lies in between the two unit quaternions q_0, q_1 along a circular arc and makes an angle $t\phi$ with q_0 and $(1-t)\phi$ with q_1, where ϕ is the angle betwen q_0 and q_1.

Alternatively, we could use unit quaternions q_0, q_1 to represent these two rotations. The quaternion generated by linear interpolation

$$q(t) = (1 - t)q_0 + t\,q_1$$

is no longer a unit quaternion, but we can adjust the length of $q(t)$ by dividing by $|q(t)|$. Now $q(t)/|q(t)$ is a unit quaternion representing an intermediate rotation. Unfortunately, the tumbling motions generated by this approach will not appear smooth and natural because linear interpolation does not generate uniformly spaced quaternions—that is, if ϕ is the angle between q_0 and q_1, then $t\phi$ is not generally the angle between q_0 and $q(t)/|q(t)|$. To solve this problem, we shall apply spherical linear interpolation (SLERP) to find the unit quaternion $q(t)$ in the plane of q_0 and q_1 that makes an $t\phi$ angle with q_0 and $(1 - t)\phi$ with q_1 (see Figure 9.1).

Theorem 9.1: *Spherical linear interpolation—SLERP*
Let q_0, q_1 be two unit quaternions and *let ϕ be the angle between q_0* and q_1. *Then the quaternion*

$$q(t) = \frac{\sin\big((1-t)\phi\big)}{\sin(\phi)}q_0 + \frac{\sin(t\phi)}{\sin(\phi)}q_1$$

is the unit quaternion in the plane of q_0, q_1 that makes the angle $t\phi$ with q_0 and $(1-t)\phi$ with q_1.
Proof: Let $q(t)$ be the unit quaternion in the plane of q_0, q_1 that makes the angle $t\phi$ with q_0 and $(1-t)\phi$ with q_1 (see Figure 9.1). Since $q(t)$ is in the plane of q_0, q_1, there are scalars $\alpha(t), \beta(t)$ such that

$$q(t) = \alpha(t)q_0 + \beta(t)q_1.$$

Moreover, since $q_0, q_1, q(t)$ are unit quaternions, dotting both sides of this equation with q_0 and q_1 yields

$$\cos(t\phi) = \alpha(t) + \cos(\phi)\beta(t)$$
$$\cos((1-t)\phi) = \cos(\phi)\alpha(t) + \beta(t).$$

Solving these two equations for $\alpha(t)$, $\beta(t)$, we find that

$$\alpha(t) = \frac{\det\begin{pmatrix} \cos(t\phi) & \cos(\phi) \\ \cos\big((1-t)\phi\big) & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & \cos(\phi) \\ \cos(\phi) & 1 \end{pmatrix}} = \frac{\cos(t\phi) - \cos(\phi)\cos\big((1-t)\phi\big)}{1-\cos^2(\phi)},$$

so

$$\begin{aligned}\alpha(t) &= \frac{\cos(t\phi) - \cos(\phi)\big(\cos(\phi)\cos(t\phi) + \sin(\phi)\sin(t\phi)\big)}{\sin^2(\phi)} \\ &= \frac{\big(1-\cos^2(\phi)\big)\cos(t\phi) - \sin(\phi)\cos(\phi)\sin(t\phi)}{\sin^2(\phi)} = \frac{\sin(\phi)\cos(t\phi) - \cos(\phi)\sin(t\phi)}{\sin(\phi)} \\ &= \frac{\sin\big((1-t)\phi\big)}{\sin(\phi)}.\end{aligned}$$

Similarly,

$$\beta(t) = \frac{\det\begin{pmatrix} 1 & \cos(t\phi) \\ \cos(\phi) & \cos\big((1-t)\phi\big) \end{pmatrix}}{\det\begin{pmatrix} 1 & \cos(\phi) \\ \cos(\phi) & 1 \end{pmatrix}} = \frac{\cos\big((1-t)\phi\big) - \cos(\phi)\cos(t\phi)}{1-\cos^2(\phi)},$$

so

$$\beta(t) = \frac{\cos(\phi)\cos(t\phi) + \sin(\phi)\sin(t\phi) - \cos(\phi)\cos(t\phi)}{\sin^2(\phi)} = \frac{\sin(t\phi)}{\sin(\phi)}.$$ ♦

We adopt the notation

$$slerp(q_0, q_1, t) = \frac{\sin\big((1-t)\phi\big)}{\sin(\phi)} q_0 + \frac{\sin(t\phi)}{\sin(\phi)} q_1, \tag{9.1}$$

where ϕ is the angle between q_0 and q_1, and we call this result *spherical linear interpolation* because we are interpolating uniformly along a circular arc instead of along a straight line. Thus by Theorem 9.1 spherical linear interpolation (*slerp*) applied to two unit quaternions generates the appropriate unit quaternions to represent intermediate rotations for key frame animation.

Exercise:

9.1. Consider two unit vectors v_0, v_1. Let ϕ be the angle between v_0 and v_1 and let $w = \dfrac{v_0 \times v_1}{|v_0 \times v_1|}$.

a. Show that

i. $slerp(v_0, v_1, t) = e^{t\phi w/2} v_0 e^{-t\phi w/2}$

ii. $v_1^* v_0 = e^{\phi w}$

b. Conclude from part a that

$$slerp(v_0, v_1, t) = (v_1^* v_0)^{t/2} v_0 (v_0^* v_1)^{t/2}.$$

• • • •

CHAPTER 10

Summary—Formulas From Quaternion Algebra

Quaternions are complex numbers on steroids. Complex numbers are vectors in the plane; quaternions are vectors in the 4-dimensional space of mass-points. Multiplication with unit complex numbers rotates vectors in the plane; multiplication with unit quaternions rotates vectors in four dimensions.

Classically, the main application of quaternions in computer graphics is to use sandwiching with unit quaternions to rotate vectors in three dimensions. The advantages of unit quaternions over 3×3 rotation matrices are that quaternions can

i. increase speed and reduce storage;
ii. avoid distortions arising from numerical inaccuracies introduced by floating point computations;
iii. interpolate easily between two rotations for key frame animation.

One of the themes of this monograph is that sandwiching with unit quaternions can also be applied effectively to compute perspective projections.

We have used a good deal of algebra and geometry in our study of quaternions. Our main geometric insights are summarized in Chapter 7. Therefore, in the remainder of this section, we shall collect for easy reference all of the important algebraic formulas we have encountered during our investigations of quaternions.

Quaternion Representations

$q = m_q O + v_q$ (coordinate free)

- m_q is a scalar mass
- O represents the origin of the points in 3-dimensional affine space
- v_q is a vector in three dimensions.

$q = (q_1, q_2, q_3, q_4)$ (coordinate dependent)

- $m_q = q_4$
- $v_q = (q_1, q_2, q_3)$

Quaternion Multiplication

- $pq = (m_p O + v_p)(m_q O + v_q) = (m_p m_q - v_p . v_q)\, O + (m_p v_q + m_q v_p + v_p \times v_q)$
- $i^2 = j^2 = k^2 = -O$ and $ij = k = -j\, i, jk = i = -kj,\ ki = j = -i\, k.$
- $pq = (p_4 q_4 - p_1 q_1 - p_2 q_2 - p_3 q_3) O + (p_4 q_1 + p_1 q_4 + p_2 q_3 - p_3 q_2)\, i$
  $+ (p_4 q_2 + p_2 q_4 + p_3 q_1 - p_1 q_3) j + (p_4 q_3 + p_3 q_4 + p_1 q_2 - p_2 q_1) k$.
- $q\, p = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} * \begin{pmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}$
- $p\, q = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \end{pmatrix} * \begin{pmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ q_1 & q_2 & q_3 & q_4 \end{pmatrix}$

Vector Products

- $uv = (-u \cdot v) O + u \times v$
- $|u| = 1 \Rightarrow u^2 = -O$
- $u \perp v \Rightarrow u\, v = u \times v$
- $(u \cdot v) O = \dfrac{-(v\, u + u\, v)}{2}$
- $u \times v = \dfrac{u\, v - v\, u}{2}$.

Exponentiation

- $e^q = O + q + \dfrac{q^2}{2!} + \dfrac{q^3}{3!} + \cdots$
- $e^{\theta N} = cos(\theta) O + sin(\theta) N$

Conjugation

- $q = q_4 O + q_1 i + q_2 j + q_3 k \Rightarrow q^* = q_4 O - q_1 i - q_2 j - q_3 k$
- $q(u, \theta) = \cos(\theta) O + \sin(\theta) u \Rightarrow q^*(u, \theta) = \cos(\theta) O - \sin(\theta) u = q(u, -\theta)$
- $(pq)^* = q^* p^*$

Length

- $|\, q\, |^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2$
- $|q|^2 = q\, q^*$
- $|p\, q| = |p|\, |q|$

Spherical Linear Interpolation

$$slerp(q_0, q_1, t) = \frac{\sin((1-t)\phi)}{\sin(\phi)} q_0 + \frac{\sin(t\phi)}{\sin(\phi)} q_1, \text{ where } \phi \text{ is the angle between } q_0 \text{ and } q_1.$$

Rotation—around the line through the origin O parallel to the unit direction vector N through the angle θ

$$q(N,\theta/2) = \cos(\theta/2)O + \sin(\theta/2)N$$
$$S_{q(N,\theta/2)}(v) = q(N,\theta/2)v\, q^*(N,\theta/2)$$
$$S_{q(N,\theta/2)}(P) = q(N,\theta/2)P\, q^*(N,\theta/2)$$

Mirror image—in the plane through the point O perpendicular to the unit normal N

$$T_N(v) = N\, v\, N$$
$$T_N(P) = N(P - 2O)N$$

Perspective projection—from the canonical eye point $E(N,\theta) = O + (\cot(\theta) - \csc(\theta))N$ *into the canonical plane* $S(N, \theta)$ *through the point* $O + \cot(\theta)N$ *perpendicular to the unit normal vector N*

$$q(N, -\theta/2) = \cos(\theta/2)O - \sin(\theta/2)N$$
$$T_{q(N,-\theta/2)}(P - E) = q(N, -\theta/2)(P - E)q(N, -\theta/2)$$

Perspective projection—from an arbitrary eye point E into an arbitrary plane S perpendicular to the unit normal vector N at a distance $d = \csc(\theta)$ *along N from the eye point E, translated by* $E(N,\theta) - E$ *into the canonical plane* $S(N,\theta)$

$$q(N, -\theta/2) = \cos(\theta/2)O - \sin(\theta/2)N$$
$$T_{q(N,-\theta/2)}(P - E) = q(N, -\theta/2)(P - E)q(N, -\theta/2)$$

Conversion between unit quaternions $q(N,\theta/2) = (q_1, q_2, q_3, q_4)$ and 4×4 *rotation matrices* $Rot(N,\theta) = (R_{i,j})$

$$Rot(N,\theta) = \begin{pmatrix} q_4^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 + 2q_3q_4 & 2q_1q_3 - 2q_2q_4 & 0 \\ 2q_1q_2 - 2q_3q_4 & q_4^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_1q_4 & 0 \\ 2q_1q_3 + 2q_2q_4 & 2q_2q_3 - 2q_1q_4 & q_4^2 - q_1^2 - q_2^2 + q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Rot(N,\theta) =$$
$$\begin{pmatrix} \cos(\theta) + (1-\cos(\theta))N_1^2 & (1-\cos(\theta))N_1N_2 + \sin(\theta)N_3 & (1-\cos(\theta))N_1N_3 - \sin(\theta)N_2 & 0 \\ (1-\cos(\theta))N_1N_2 - \sin(\theta)N_3 & \cos(\theta) + (1-\cos(\theta))N_2^2 & (1-\cos(\theta))N_2N_3 + \sin(\theta)N_1 & 0 \\ (1-\cos(\theta))N_1N_3 + \sin(\theta)N_2 & (1-\cos(\theta))N_2N_3 - \sin(\theta)N_1 & \cos(\theta) + (1-\cos(\theta))N_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$q(N,\theta)=\frac{\sqrt{R_{1,1}+R_{2,2}+R_{3,3}+1}}{2}\,O+\frac{(R_{2,3}-R_{3,2})i+(R_{3,1}-R_{1,3})j+(R_{1,2}-R_{2,1})k}{2\sqrt{R_{1,1}+R_{2,2}+R_{3,3}+1}}\quad \theta\neq\pi.$$

$$q(N,\theta)=\frac{\sqrt{R_{1,1}-R_{2,2}-R_{3,3}+1}}{2}\,i+\frac{(R_{1,2}+R_{2,1})j+(R_{1,3}+R_{3,1})k+(R_{3,2}-R_{2,3})O}{2\sqrt{R_{1,1}-R_{2,2}-R_{3,3}+1}}\quad \theta\approx\pi$$

$$q(N,\theta)=\frac{\sqrt{-R_{1,1}+R_{2,2}-R_{3,3}+1}}{2}\,j+\frac{(R_{1,2}+R_{2,1})i+(R_{2,3}+R_{3,2})k+(R_{3,1}-R_{1,3})O}{2\sqrt{-R_{1,1}+R_{2,2}-R_{3,3}+1}}\quad \theta\approx\pi$$

$$q(N,\theta)=\frac{\sqrt{-R_{1,1}-R_{2,2}+R_{3,3}+1}}{2}\,k+\frac{(R_{1,3}+R_{3,1})i+(R_{2,3}+R_{3,2})j+(R_{1,2}-R_{2,1})O}{2\sqrt{-R_{1,1}-R_{2,2}+R_{3,3}+1}}\quad \theta\approx\pi$$

Conversion between unit vectors $N=(N_1,N_2,N_3,0)$ and 4×4 *mirror image matrices* $Mir(N)=(M_{i,j})$

$$Mir(N)=\begin{pmatrix}1-2N_1^2 & -2N_1N_2 & -2N_1N_3 & 0\\ -2N_1N_2 & 1-2N_2^2 & -2N_2N_3 & 0\\ -2N_1N_3 & -2N_2N_3 & 1-2N_3^2 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$N_k=\sqrt{\frac{1-M_{kk}}{2}},\ k=1,2,3.$$

Conversion between unit quaternions $q(N,-\theta/2)=(q_1,q_2,q_3,q_4)$ and 4×4 *perspective projection matrices* $Persp(N,\theta)=(P_{i,j})$

$$Persp(N,\theta)=\begin{pmatrix}1-2q_1^2 & -2q_1q_2 & -2q_1q_3 & -2q_1q_4\\ -2q_1q_2 & 1-2q_2^2 & -2q_2q_3 & -2q_2q_4\\ -2q_1q_3 & -2q_2q_3 & 1-2q_3^2 & -2q_3q_4\\ 2q_1q_4 & 2q_2q_4 & 2q_3q_4 & 2q_4^2-1\end{pmatrix}$$

$$Persp(N,\theta)=\begin{pmatrix}1+(\cos(\theta)-1)N_1^2 & (\cos(\theta)-1)N_1N_2 & (\cos(\theta)-1)N_1N_3 & \sin(\theta)N_1\\ (\cos(\theta)-1)N_1N_2 & 1+(\cos(\theta)-1)N_2^2 & (\cos(\theta)-1)N_2N_3 & \sin(\theta)N_2\\ (\cos(\theta)-1)N_1N_3 & (\cos(\theta)-1)N_2N_3 & 1+(\cos(\theta)-1)N_3^2 & \sin(\theta)N_3\\ -\sin(\theta)N_1 & -\sin(\theta)N_2 & -\sin(\theta)N_3 & \cos(\theta)\end{pmatrix}$$

$$Persp(N,d) = \begin{pmatrix} 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_1^2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_3 & N_1/d \\ \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_2 & 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_2^2 & \left(\pm\sqrt{1-1/d^2}-1\right)N_2N_3 & N_2/d \\ \left(\pm\sqrt{1-1/d^2}-1\right)N_1N_3 & \left(\pm\sqrt{1-1/d^2}-1\right)N_2N_3 & 1+\left(\pm\sqrt{1-1/d^2}-1\right)N_3^2 & N_3/d \\ -N_1/d & -N_2/d & -N_3/d & \pm\sqrt{1-1/d^2} \end{pmatrix}$$

- $d = \csc(\theta) =$ distance from eye to plane of projection
- $E(N,\theta) = O + \left(\pm\sqrt{d^2-1} - d\right)N =$ eye point
- $S(N,\theta) =$ plane of projection through the point $Q = O \pm \sqrt{d^2-1}\,N$ perpendicular to the unit normal vector N

$$q_4 = \sqrt{\frac{P_{44}+1}{2}} \text{ and } q_k = \frac{P_{4k}-P_{k4}}{4q_4},\ k = 1,2,3.$$

PART III

Rethinking Quaternions and Clifford Algebras

CHAPTER 11

Goals and Motivation

Clifford algebras are introduced into mathematics to fill a gap in the algebra of vector spaces. Complex numbers allow us to multiply vectors in the plane, and quaternions allow us to multiply vectors in four dimensions. These products facilitate the study of transformations such as rotations in dimensions 2, 3, and 4. Unfortunately, there are no such rules for multiplication of vectors in higher dimensions—that is, there are no products in dimensions greater than four that are associative, distribute through addition, and most crucially, where every nonzero vector has an inverse for multiplication so that we can solve linear equations (see Section 4, especially Exercises 4.9–4.12). This void makes the study of transformations in higher dimensions more cumbersome than in lower dimensions; typically we need to resort to matrix methods rather than rely on direct manipulation of vector products.

Clifford algebra associates to each n-dimensional vector space an algebra of dimension 2^n—that is, a vector space of dimension 2^n where not only the sum, but also the product of every two elements in this algebra is defined. As usual multiplication is associative and distributes through addition, but not all nonzero elements in this algebra have inverses for multiplication. Nevertheless, Clifford algebras seem to be the natural generalizations of complex numbers and quaternions to higher dimensional vector spaces. Like complex numbers and quaternions, Clifford algebras facilitate the study of transformations such as rotations in higher dimensions. Since complex numbers and quaternions play the same role in low dimensions, we would expect there to be a close connection in low dimensions between Clifford algebras and both complex numbers and quaternions. The purpose of Part III is to investigate these connections as well as to show how our previous insights regarding quaternions can be applied in Clifford algebras.

The ambient space of contemporary computer graphics is the 4-dimensional vector space of mass-points. A homogeneous model for Clifford algebra that deals directly with mass-points has indeed been constructed, but this model is actually a Clifford algebra for R^5 [Dorst *et al.*, 2007]. Thus this algebra is a vector space with $2^5 = 32$ dimensions. This 32-dimensional algebra has many convenient properties. For example, not only lines and planes but also circles and spheres are included as primitive objects in this homogeneous model; moreover all conformal transformations on R^3 including inversions in the sphere are represented in this algebra. Nevertheless, the dimension of this algebra seems excessively high and the algebra itself unnecessarily complicated. One goal of Part III

is to develop a simpler, lower dimensional Clifford algebra for investigating the 4-dimensional space of mass-points.

We shall construct a homogeneous model for computer graphics using the 8-dimensional Clifford algebra for R^3. To incorporate points as well as vectors within this model, we employ the odd dimensional elements of this graded 8-dimensional algebra to represent mass-points by exploiting the pseudoscalars to represent mass. The even dimensional elements of this Clifford algebra are isomorphic to the quaternions, which operate on the odd dimensional elements by sandwiching. As with quaternions, one of the major benefits of this model is that along with the standard sandwiching formulas in Clifford algebra for rotations and reflections, this new paradigm also allows us to use sandwiching to compute perspective projections.

Our goal is to make four principal contributions to the theory of Clifford algebras:

1. To show how to apply the standard Clifford algebra for R^3 to represent mass-points by taking advantage of the pseudoscalars to represent mass;
2. To present better ways to visualize the effect of the sandwiching operation on points and vectors in this Clifford algebra based on insights from our study of quaternions;
3. To develop simple, intuitive proofs of the sandwiching formulas for rotations and reflections in this Clifford algebra based on insights from our study of quaternions;
4. To show that perspective projections can also be incorporated into the standard framework of transformations represented in this Clifford algebra.

Part III is organized in the following fashion. We begin in Chapter 12 with a brief description of the general Clifford algebra for n-dimensional vector spaces, and we observe that for $n \geq 2$ each of these Clifford algebras contain the quaternions as a subalgebra. The Clifford algebra for $\boldsymbol{R}^2$ is actually isomorphic to the quaternions, so in Chapter 13 we investigate the Clifford algebra for $\boldsymbol{R}^2$ and we show how this Clifford algebra is used to study conformal maps on $\boldsymbol{R}^2$. In Chapter 14 we provide a brief review for the uninitiated of the standard objects—scalars, vectors, bivectors, and pseudoscalars—along with some of the basic formulas—Clifford product, wedge product, duality—for the Clifford algebra of R^3. Readers already familiar with Clifford algebra can skip this tutorial. In Chapter 15 we introduce the operators and operands—mass-points and quaternions—by exploiting the pseudoscalars to represent mass. Chapter 16 is devoted to studying the action of the unit quaternions on the space of mass-points, and Chapter 17 shows how to apply the sandwiching operators to compute rotation, mirror image, and perspective projection on points as well as on vectors. Many of these results will already be known to readers familiar with Clifford algebra, but the material on perspective projection in Section 17.3 appears to be completely new. We summarize our principal insights and results in Chapter 18, and we close in Chapter 19 with some simple, alternative approaches to a homogeneous model for computer graphics.

CHAPTER 12

Clifford Algebras and Quaternions

Clifford algebra associates to each n-dimensional vector space $\boldsymbol{R}^n$ a geometric algebra of dimension 2^n—that is, a vector space of dimension 2^n where not only the sum, but also the product of every two elements in the algebra is defined. Let $e_1,\ldots,e_n$ be an orthonormal basis for $\boldsymbol{R}^n$. Then the 2^n canonical generators (basis vectors) of the Clifford algebra for $\boldsymbol{R}^n$ are denoted by the products:

1	*scalars*
$e_1,\ldots,e_n$	*vectors*
$e_1e_2,\ldots,e_{n-1}e_n$	*bivectors*
$\vdots$	$\vdots$
$e_1\cdots e_k,\ldots,e_{n-k+1}\cdots e_n$	*k-vectors*
$\vdots$	$\vdots$
$e_1\cdots e_n$	*pseudoscalars*

Notice that there are $\binom{n}{k}$ products with exactly k factors, so there are a total of 2^n products.

The formal algebra of this 2^n-dimensional vector space is defined by the following rules:

i. multiplication is associative;
ii. multiplication distributes through addition;
iii. 1 is the identity for multiplication;
iv. $e_k^2 = \pm 1;$ *(12.1)*
v. $e_j\, e_i = -e_i\, e_j. \qquad j \neq i$ *(12.2)*

It follows from Equations (12.1) and (12.2) that every product of the basis vectors $e_1,\ldots,e_n$ with more than n factors can be rewritten, up to sign, as one of the 2^n canonical generators.

The minus signs on the right-hand side of Equation (12.2) are there to capture orientation: the orientation of the vectors e_j,e_i is opposite to the orientation of the vectors e_i,e_j. In contrast, the signs in Equation (12.1) may depend on k, and different geometries may be investigated with different choices of sign. We shall see in Chapters 13 and 14 that for our purposes it is convenient to choose these sign to be negative for all k so that

$$e_k^2 = -1 \,. \tag{12.3}$$

The simplest Clifford algebra is the Clifford algebra associated with the one-dimensional vector space $\boldsymbol{R}$. This 2-dimensional vector space is spanned by the two element $1, e_1$. Since $e_1^2 = -1$, this Clifford algebra for $\boldsymbol{R}$ is isomorphic to the complex numbers $\boldsymbol{C}$.

The Clifford algebra for $\boldsymbol{R}^2$ is the 4-dimensional algebra generated by the canonical basis $1, e_1, e_2, e_1 e_2$. If we let

$$i = e_1, \quad j = e_2, \quad k = e_1 e_2 \tag{12.4}$$

then by Equations (12.2) and (12.3) it is easy to verify that

$$i^2 = j^2 = k^2 = -1 \tag{12.5}$$

$$i\,j = k, \quad j\,k = i, \quad k\,i = j \tag{12.6}$$

which is the standard algebra of quaternion multiplication. Thus the Clifford algebra for $\boldsymbol{R}^2$ is isomorphic to the algebra of quaternions. Nevertheless, as we shall see in Chapter 13, this Clifford algebra is typically used not to model rotations in $\boldsymbol{R}^4$, but rather to model rotations in $\boldsymbol{R}^2$.

The Clifford algebra for $\boldsymbol{R}^3$ contains the quaternions as a subalgebra. This 8-dimensional algebra is generated by the canonical basis

$$1,\ e_1,\ e_2,\ e_3,\ e_1e_2,\ e_2e_3,\ e_3e_1,\ e_1e_2e_3\,.$$

If we let

$$i = e_1 e_2, \quad j = e_2 e_3, \quad k = e_3 e_1 \tag{12.7}$$

then using Equations (12.2) and (12.3) it is again easy to verify that

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ ij = k, jk = i, ki = j \end{gathered} \tag{12.8}$$

In fact, by the same construction the Clifford algebra for $\boldsymbol{R}^n$ for every $n \geq 3$ contains the quaternions as a subalgebra.

Typically the Clifford algebra for $\boldsymbol{R}^n$ is used to study transformations on $\boldsymbol{R}^n$. Thus transformations on the plane are typically modeled by the Clifford algebra for $\boldsymbol{R}^2$, and transformations on $\boldsymbol{R}^3$ are typically modeled by the Clifford algebra for $\boldsymbol{R}^3$. We are going to investigate each of these Clifford algebras in turn to appreciate more precisely the role of quaternions in modeling transformations using Clifford algebra.

CHAPTER 13

Clifford Algebra for the Plane

The Clifford algebra for the plane $\boldsymbol{R}^2$ is the 4-dimensional algebra generated by the canonical basis $1, e_1, e_2, e_1 e_2$. Recall that this algebra is isomorphic to the quaternions, since by Equations (12.2) and (12.3)

$$e_1^2 = e_2^2 = (e_1 e_2)^2 = -1$$
$$(e_1 e_2) e_1 = e_2 \text{ and } e_2 (e_1 e_2) = e_1 .$$

Let

$$I = e_1 e_2; \tag{13.1}$$

then

$$I^2 = -1. \tag{13.2}$$

Thus the subalgebra generated by the two elements $1, I$ is isomorphic to the complex numbers. We shall apply this subalgebra to model transformations on the plane of vectors generated by the canonical basis vectors e_1, e_2.

We use the vectors spanned by e_1, e_2 to model vectors in the plane, and we think of e_1, e_2 as representing unit vectors along the x and y-axes. We use the elements spanned by 1, I, which we think of as complex numbers, to model transformations on these vectors in the plane. For example, by Equations (12.2) and (12.3),

$$\begin{aligned} I e_1 &= e_2 \\ I e_2 &= -e_1. \end{aligned} \tag{13.3}$$

Thus multiplying the basis vectors on the left by I rotates the basis vectors by $\pi/2$. Similarly,

$$\begin{aligned} e_1 I &= -e_2 \\ e_2 I &= e_1. \end{aligned} \tag{13.4}$$

Thus multiplying the basis vectors on the right by I rotates the basis vectors by $-\pi/2$ More generally we have the following results.

Theorem 13.1: *(The action of complex numbers on vectors in the plane)*
Let

$$u(\theta) = \cos(\theta)e_1 + \sin(\theta)e_2$$

denote a unit vector in the plane, and *let*

$$z(\phi) = \cos(\phi) + \sin(\phi)I$$

be a unit complex number. Then

$$\text{i. } z(\phi)u(\theta) = u(\theta+\phi)$$
$$\text{ii. } u(\theta)z(\phi) = u(\theta-\phi)$$

Proof: By Equation (13.3)

$$\begin{aligned} z(\phi)u(\theta) &= (\cos(\phi) + \sin(\phi)I)(\cos(\theta)e_1 + \sin(\theta)e_2) \\ &= (\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta))e_1 + (\cos(\phi)\sin(\theta)+\sin(\phi)\cos(\theta))e_2 \\ &= \cos(\theta+\phi)e_1 + \sin(\theta+\phi)e_2 = u(\theta+\phi) \end{aligned}$$

Similarly, by Equation (13.4)

$$\begin{aligned} u(\theta)z(\phi) &= (\cos(\theta)e_1 + \sin(\theta)e_2)(\cos(\phi) + \sin(\phi)I) \\ &= (\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi))e_1 + (\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi))e_2 \\ &= \cos(\theta-\phi)e_1 + \sin(\theta-\phi)e_2 = u(\theta-\phi). \end{aligned}$$

♦

Corollary 13.2: *(The action of complex numbers on vectors in the plane)*
Multiplying a vector u by the complex number $\mathbf{z}(\phi)=\cos(\phi)+\sin(\phi)I$ *on the left rotates the vector counter-clockwise by the angle* ϕ. *Similarly, multiplying a vector u by the complex number* $\mathbf{z}(\phi)=\cos(\phi)+\sin(\phi)I$ *on the right rotates the vector clockwise by the angle* ϕ.

These rotations in opposite directions depending on whether we perform multiplication on the left or the right should resonate from our study of quaternion multiplication, since these for-

mulas are precisely the results we found in Chapter 5 for the geometry of quaternion multiplication. Moreover we also have the following familiar result for composing transformations represented by complex numbers.

Proposition 13.3 *(The product of complex numbers)*
Let

$$\boldsymbol{z}(\phi_1) = \cos(\phi_1) + \sin(\phi_1) I$$
$$\boldsymbol{z}(\phi_2) = \cos(\phi_2) + \sin(\phi_2) I$$

be a pair of complex number. Then

i. $\boldsymbol{z}(\phi_1)\, \boldsymbol{z}(\phi_2) = \boldsymbol{z}(\phi_1 + \phi_2)$.
ii. $\boldsymbol{z}(\phi_1)\, \boldsymbol{z}(\phi_2) = \boldsymbol{z}(\phi_2)\, z(\phi_1)$.

Proof: By Equation (13.2)

i. $$\begin{aligned}\boldsymbol{z}(\phi_1)\, \boldsymbol{z}(\phi_2) &= (\cos(\phi_1) + \sin(\phi_1)\, I)(\cos(\phi_2) + \sin(\phi_2) I) \\ &= (\cos(\phi_1)\cos(\phi_2) - \sin(\phi_1)\sin(\phi_2)) + (\cos(\phi_1)\sin(\phi_2) + \sin(\phi_1)\cos(\phi_2)) I \\ &= \cos(\phi_1 + \phi_2) + \sin(\phi_1 + \phi_2) I = \boldsymbol{z}(\phi_1 + \phi_2).\end{aligned}$$

ii. Follows immediately from part i. ♦

Corollary 13.4: *(Composing transformations in the plane)*
The product of two complex numbers represents the composite of the transformations represented by each of these individual complex numbers.

By Corollaries 13.3 and 13.4 multiplication with complex numbers rotates vectors in the real plane $\boldsymbol{R}^2$ in opposite directions, depending on whether the multiplication is on the left or the right, whereas multiplication by complex numbers rotates vectors in the complex plane $\boldsymbol{C}$ in same directions, independent of whether the multiplication is on the left or the right. In Chapter 5 we harnessed these two distinct quaternion planes (real and complex) to represent vectors in $\boldsymbol{R}^4$, and we took advantage of the different effects of multiplication by complex numbers in these two mutually orthogonal planes (double isoclinic rotations—see Figure 5.3) to invoke sandwiching to model rotation, reflection, and perspective projection in $\boldsymbol{R}^3$. In contrast, the Clifford algebra approach is to use the Clifford algebra for $\boldsymbol{R}^2$ (the quaternions) to study transformations on $\boldsymbol{R}^2$.

Notice that both rotation and scaling can be represented by multiplying vectors in $\boldsymbol{R}^2$ by complex numbers, since

$$(sz(\phi))u = z(\phi)(su).$$

Thus any conformal transformation of vectors in the plane can be represented in this way by multiplication with a single complex number.

The main innovation of the Clifford algebra approach to transformations of vectors in the plane is to separate the operators and the operands—that is, to separate the vectors in the real plane from the operators in the complex plane. In contrast, when we use complex numbers to represent both vectors in the plane and transformations on these vectors, the complex numbers play two roles: as operators and as operands. These dual roles confound operators and operands, and often it is difficult to tell if a particular complex number represents an operator or an operand. In Clifford algebra there is no such ambiguity, but the price we pay for this conceptual clarity is to adopt a 4-dimensional algebra (the quaternions) to replace a 2-dimensional algebra (the complex numbers).

Exercises:

13.1. Prove that in the Clifford algebra for R^2

$$e^{\phi I} = \cos(\phi) + \sin(\phi) I.$$

13.2. Let e_1, e_2 and e'_1, e'_2 be two orthonormal bases for R^2, and let $I = e_1 e_2$ and $I' = e'_1 e'_2$.

a. Show that $I' = I$.

b. Conclude from part *a* that the complex number I is independent of the choice of the orthonormal basis for R^2.

CHAPTER 14

The Standard Model of the Clifford Algebra for Three Dimensions

We present here a brief review of the basic algebra underlying the standard model for the Clifford algebra of a real 3-dimensional vector space.

14.1 SCALARS, VECTORS, BIVECTORS, AND PSEUDOSCALARS

The Clifford algebra associated with the 3-dimensional vector space R^3 is an 8-dimensional vector space. Let e_1, e_2, e_3 be an orthonormal basis for R^3. Then the eight canonical generators for the Clifford algebra of R^3 are denoted by the products:

$$1,\ e_1,\ e_2,\ e_3,\ e_1e_2,\ e_2e_3,\ e_3e_1,\ e_1e_2e_3.$$

The formal algebra of this 8-dimensional vector space is defined, as usual, by the following rules:

i. multiplication is associative;
ii. multiplication distributes through addition;
iii. 1 is the identity for multiplication;
iv. $e_1^2 = e_2^2 = e_3^2 = -1;$ *(14.1)*
v. $e_2e_1 = -e_1e_2,\ e_3e_2 = -e_2e_3,\ e_1e_3 = -e_3e_1.$ *(14.2)*

It follows from Equations (14.1) and (14.2) that every product of the basis vectors e_1, e_2, e_3 with four of more elements can be rewritten, up to sign, as one of the eight generators.

The sign on the right-hand side of Equation (14.1) is somewhat arbitrary. The minus sign is chosen here for convenience for three reasons:

i. so that $(e_1e_2e_3)^2 = 1$ rather than -1—see Section 14.3, Equation (14.17)
ii. so that the formula for the product of two vectors more closely mimics that standard formula for quaternion multiplication—see Section 14.2, Equation (14.8)
iii. so that the formulas for duality are expressed by multiplication with $-e_1e_2e_3$—see Section 14.3.

Multiples of the element 1 are called *scalars*; multiples of the generator $e_1\, e_2\, e_3$ are called *pseudoscalars*. Linear combinations of e_1, e_2, e_3 are called *vectors*; linear combinations of e_1e_2, e_2e_3, e_3e_1 are called *bivectors*. Thus the scalars and pseudoscalars each form 1-dimensional subspaces, and the vectors and bivectors each form 3-dimensional subspaces of the Clifford algebra for R^3.

This Clifford algebra is also a graded algebra. The dimension of a scalar is 0; the dimension of a vector is 1; the dimension of a bivector is 2; and the dimension of a pseudoscalar is 3. Geometrically, we typically think of a vector as representing a length in a particular direction. Similarly, we will see that we shall think of a bivector as representing an area in a specific plane. The pseudoscalars are commonly used to model volume elements, but contrary to common usage, we are going to use the pseudoscalars to represent mass at a point rather than volume in three-space (see Section 15.1).

14.2 WEDGE PRODUCT AND CROSS PRODUCT

Consider the Clifford product of two arbitrary vectors: $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$. Since multiplication distributes through addition, it follows from Equations (14.1) and (14.2) that

$$\begin{aligned} uv = (u_1e_1 + u_2e_2 + u_3e_3)(v_1e_1 + v_2e_2 + v_3e_3) &= -(u_1v_1 + u_2v_2 + u_3v_3) \\ &+ (u_1v_2 - u_2v_1)e_1e_2 + (u_2v_3 - u_3v_2)e_2e_3 + (u_3v_1 - u_1v_2)e_3e_1. \end{aligned} \tag{14.3}$$

To shorten this expression for the Clifford product uv, we define the *wedge product* $u \wedge v$ of two vectors u, v to be the bivector given by setting

$$u \wedge v = (u_1v_2 - u_2v_1)e_1e_2 + (u_2v_3 - u_3v_2)e_2e_3 + (u_3v_1 - u_1v_3)e_3e_1. \tag{14.4}$$

By Equations (14.2)–(14.4), the wedge product is anticommutative and distributes through addition—that is

$$v \wedge u = -u \wedge v \tag{14.5}$$

$$u \wedge (v + w) = u \wedge v + u \wedge w. \tag{14.6}$$

It follows immediately from Equation (14.5) that for any vector v,

$$v \wedge v = 0. \tag{14.7}$$

Using the wedge product and applying Equations (14.3) and (14.4), we now have a simple expression for the Clifford product of two vectors u, v:

$$uv = -u \cdot v + u \wedge v. \tag{14.8}$$

Thus the Clifford product uv of two vectors u, v is the sum of a scalar $(-u \cdot v)$ and a bivector $(u \wedge v)$. Notice the similarity to the quaternion product of two vectors in Chapter 4 (Equation 4.6). Notice too that since the basis vectors e_1, e_2, e_3 are orthogonal

$$e_1 e_2 = e_1 \wedge e_2, \; e_2 e_3 = e_2 \wedge e_3, \; e_3 e_1 = e_3 \wedge e_1. \tag{14.9}$$

In general, the Clifford product uv is neither commutative nor anticommutative, since

$$v \cdot u = u \cdot v$$

but

$$v \wedge u = -u \wedge v.$$

The wedge product of two vectors is closely related to their cross product. Recall that e_1, e_2, e_3 is an orthonormal basis for R^3, so

$$\begin{aligned} u \times v &= (u_1 v_2 - u_2 v_1) e_1 \times e_2 + (u_2 v_3 - u_3 v_2) e_2 \times e_3 + (u_3 v_1 - u_1 v_3) e_3 \times e_1. \\ &= (u_1 v_2 - u_2 v_1) e_3 + (u_2 v_3 - u_3 v_2) e_1 + (u_3 v_1 - u_1 v_3) e_2 \end{aligned} \tag{14.10}$$

Thus the wedge product and the cross product have the same scalar components, but the wedge product is a bivector whereas the cross product is a vector.

14.3 DUALITY

We can use the pseudoscalar

$$O = -e_1 e_2 e_3 \tag{14.11}$$

to convert back and forth between the wedge product and the cross product. Indeed observe that by Equations (14.1) and (14.2)

$$Oe_1 = e_2 e_3 = e_1 O, \qquad Oe_2 = e_3 e_1 = e_2 O, \qquad Oe_3 = e_1 e_2 = e_3 O \tag{14.12}$$

$$Oe_1 e_2 = e_3 = e_1 e_2 O, \qquad Oe_2 e_3 = e_1 = e_2 e_3 O, \qquad Oe_3 e_1 = e_2 = e_3 e_1 O, \tag{14.13}$$

Therefore by Equations (14.4), (14.10), (14.12), and (14.13):

$$O(u \wedge v) = u \times v = (u \wedge v) O \tag{14.14}$$

$$O(u \times v) = u \wedge v = (u \times v) O. \tag{14.15}$$

The pseudoscalar O shares many properties with the scalar 1. For example, it follows easily from Equations (14.12) and (14.13) that O commutes with every element of the Clifford algebra—that is, for any element p of the Clifford algebra,

$$Op = pO. \tag{14.16}$$

Moreover by Equations (14.1) and (14.2)

$$O^2 = 1. \tag{14.17}$$

By Equations (14.12) and (14.13), multiplication by O is a vector space isomorphism between the even dimensional elements of the Clifford algebra and the odd dimensional elements of the Clifford algebra. For any element p of the Clifford algebra, we call $Op = pO$ the *dual of p*. Thus vectors and bivectors are dual—in particular, by Equations (14.14) and (14.15) cross products and wedge products are dual—and scalars and pseudoscalars are also dual.

Exercises:

14.1. Let u be a nonzero vector in R^3.

a. Show that

$$u^{-1} = \frac{-u}{|u|^2}.$$

b. Conclude that every nonzero vector in R^3 has an inverse in the Clifford algebra for R^3.

14.2. Let e_1, e_2, e_3 be an orthonormal basis for R^3, and let $O = -e_1e_2e_3$.

a. Show that

$$(1+O)(1-O) = 0.$$

b. Conclude from part a that $1+O$ does not have an inverse in the Clifford algebra for R^3.

c. Conclude from part b that the Clifford algebra for R^3 is not a division algebra.

14.3. Consider three vectors

$$u = u_1e_1 + u_2e_2 + u_3e_3,\ v = v_1e_1 + v_2e_2 + v_3e_3, \text{ and } w = w_1e_1 + w_2e_2 + w_3e_3.$$

a. Show that

$$uvw = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} e_1e_2e_3 .$$

b. Conclude from part a that $O = -e_1 e_2 e_3$ is independent of the choice of the orthonormal basis e_1, e_2, e_3.

14.4 BIVECTORS

Bivectors are to planes as vectors are to lines. A vector v represents a length $|v|$ in a fixed direction. A bivector $u \wedge v$ represents an area $|u \wedge v| = area(u,v)$ within a fixed plane, the plane determined by the vectors u, v. The following proposition asserts that every bivector can be factored as the wedge product of two vectors. Therefore every bivector represents an area within a fixed plane.

Proposition 14.1: *Let b be a bivector. Then there are vectors u, v such that $b = u \wedge v$. Moreover we can choose u, v so that u is perpendicular to v.*

Proof: Consider an arbitrary bivector

$$b = b_1\, e_2\, e_3 + b_2\, e_3\, e_1 + \ b_3\, e_1\, e_2 = b_1\, e_2 \wedge e_3 + b_2\, e_3 \wedge e_1 + b_3\, e_1 \wedge e_2.$$

Since the wedge product is anticommutative and distributes through addition,

$$b = (b_1\, e_2 - b_2\, e_1) \wedge e_3 + b_3\, e_1 \wedge e_2 = (b_1\, e_2 - b_2\, e_1) \wedge e_3 - \frac{b_3}{b_2}(b_1\, e_2 - b_2\, e_1) \wedge e_2\,.$$

Thus

$$b = (b_1\, e_2 - b_2\, e_1) \wedge \frac{(b_2 e_3 - b_3 e_2)}{b_2},$$

so there are indeed vectors u, v such that $b = u \wedge v$. Moreover, since $v \wedge v = 0$,

$$b = \left(u - \frac{(u \cdot v)v}{v \cdot v} \right) \wedge v$$

and

$$\left(u - \frac{(u \cdot v)v}{v \cdot v} \right) \cdot v = 0 \ \Rightarrow \ \left(u - \frac{(u \cdot v)v}{v \cdot v} \right) \perp v \quad \blacklozenge$$

Corollary 14.2: *The vector bO is orthogonal to the plane determined by the bivector b.*

Proof: By Proposition 14.1, there are vectors u, v such that $b = u \wedge v$. Therefore

$$bO = (u \wedge v)O = u \times v,$$

so $b\,O = u \times v$ is orthogonal to the plane of u, v, which is the plane determined by b. ♦

By Proposition 14.1 for any bivector b, there are vectors u and v such that $b = u \wedge v$. We define that magnitude of the bivector $b = u \wedge v$ by setting

$$|b| = |u \wedge v| = area(u, v) = |u \times v|. \tag{14.18}$$

Notice that the magnitude of a bivector b is independent of the choice of the representatives u, v for the bivector b, since by Equations (14.10) and (14.15)

$$u_1 \wedge v_1 = u_2 \wedge v_2 \Leftrightarrow u_1 \times v_1 = u_2 \times v_2 \Rightarrow |u_1 \wedge v_1| = |u_1 \times v_1| = |u_2 \times v_2| = |u_2 \wedge v_2|.$$

By Equation (14.8) for every vector v,

$$v^2 = -|v|^2.$$

Next we are now going show that the same result holds for bivectors: that is, for every bivector b,

$$b^2 = -|b|^2.$$

We begin by using duality to show that bivectors multiply in a manner analogous to vectors.

Proposition 14.3: *Let b_1, b_2 be bivectors. Then*

$$b_1 b_2 = -(b_1 O) \cdot (b_2 O) + (b_1 O) \wedge (b_2 O).$$

Proof: Recall that by Equation (14.8) for all vectors u, v

$$uv = -u \cdot v + u \wedge v.$$

Therefore, since $O^2 = 1$ and O commutes with every element in the Clifford algebra,

$$b_1 b_2 = b_1 b_2 O^2 = (b_1 O)(b_2 O) = -(b_1 O) \cdot b_2 O) + (b_1 O) \wedge (b_2 O). \qquad \blacklozenge$$

Corollary 14.4: *Let b be a bivector. Then $b^2 = -|b|^2$.*
Proof: By Proposition 14.1, there are vectors u, v such that $b = u \wedge v$. Therefore by Proposition 14.3 and Equations (14.14) and (14.18)

$$b^2 = -(bO) \cdot (bO) = -|u \times v|^2 = -|b|^2. \qquad \blacklozenge$$

14.5 QUATERNIONS

The algebra generated by the even dimensional basis elements 1, e_1e_2, e_2e_3, e_3e_1 of the Clifford algebra for R^3 is isomorphic to the algebra of quaternions. Indeed let

$$i = e_2e_3, j = e_3e_1, k = e_1e_2. \tag{14.19}$$

Then by Equations (14.1) and (14.2) it is easy to verify that

$$i^2 = j^2 = k^2 = -1 \tag{14.20}$$

$$ij = k, jk = i, ki = j \tag{14.21}$$

which is the standard algebra of quaternion multiplication (see Chapter 4). We shall use the classical notation H to denote the quaternion algebra, represented here by the even dimensional elements of the Clifford algebra.

The quaternion subalgebra of the Clifford algebra is essentially identical to the algebra of quaternions presented in Chapter 4; the only difference is that for the quaternions in the Clifford algebra, vectors are replaced by bivectors. We present a brief summary of some of the highlights of this algebra below. The reader should note the obvious parallels to the discussion in Chapter 4.

Recall, for example, that conjugates play an important role in quaternion algebra. Consider a quaternion, $q = m_q + b_q$, where m_q is a scalar and b_q is a bivector. In Clifford algebra, the *conjugate* of q, denoted by q^*, is defined by

$$q^* = m_q - b_q. \tag{14.22}$$

By Corollary 14.4 and Equation (14.19)–(14.22)

$$qq^* = m_q^2 - b_q^2 = m_q^2 + |b_q|^2;$$

Define $|q|^2 = qq^*$. Then

$$q^{-1} = \frac{q^*}{|q|^2}$$

Every quaternion can be written as

$$q = m_q + b_q = m_q + n_q b'_q,$$

where n_q is a scalar, b'_q is a unit bivector, and $n_q b'_q = b_q$. If q is a unit quaternion, then

$$m_q^2 + n_q^2 = |q|^2 = 1.$$

Therefore there is an angle θ such that

$$m_q = \cos(\theta)$$

$$n_q = \sin(\theta).$$

Hence every unit quaternion can be expressed as

$$q(b',\theta)=\cos(\theta)+\sin(\theta)b', \tag{14.23}$$

where $\cos(\theta)=m_q$ is a scalar and b' is a unit bivector. Thus for unit quaternions $q(b',\theta)$, the conjugate is given by

$$q^*(b',\theta)=\cos(\theta)-\sin(\theta)b'=q(b',-\theta). \tag{14.24}$$

Finally, we have the following classical result concerning the conjugate of the product of two quaternions. This result follows from Proposition 4.4 and the isomorphism of H to the standard quaternions. But for completeness, we give the analogous Clifford algebra proof here as well.

Proposition 14.5: $(pq)^*=q^*p^*$
Proof: Let $p=m_p+b_p$ and $q=m_q+b_q$. Recall that by Proposition 14.3,

$$b_pb_q=-(b_pO)\cdot(b_qO)+(b_pO)\wedge(b_qO).$$

Therefore

$$pq=(m_p+b_p)(m_q+b_q)=m_pm_q+m_pb_q+m_qb_p+b_pb_q$$

$$=(m_pm_q-b_pO\cdot b_qO)+(m_pb_q+m_qb_p+b_pO\wedge b_qO)$$

and

$$q^*p^*=(m_q-b_q)(m_p-b_p)=m_qm_p-m_qb_p-m_pb_q+b_qb_p$$

$$=(m_qm_p-b_qO\cdot b_pO)-(m_pb_q+m_qb_p-b_qO\wedge b_pO).$$

Comparing the right-hand sides of these two equations and recalling that the wedge product is anticommutative, we conclude that

$$(pq)^*=q^*p^*. \qquad \blacklozenge$$

Exercise:

14.4. Let p, q be quaternions. Show that

$$|pq|=|p||q|.$$

CHAPTER 15

Operands and Operators—Mass-Points and Quaternions

The Clifford algebra of R^3 is a real 8-dimensional vector space. This vector space splits conveniently into two 4-dimensional subspaces: one consisting of the even dimensional elements, the quaternions H, and one consisting of the odd dimensional elements, the duals HO of the quaternions. Let

$Cl(R^3)$ = the Clifford algebra of R^3

$Cl(R^3)^{even}$ = the even dimensional elements of $Cl(R^3) \cong H$

$Cl(R^3)^{odd}$ = the odd dimensional element of $Cl(R^3) \cong HO$.

Then we have the following vector space isomorphisms:

$$\begin{array}{ccccc} Cl(R^3) & \cong & Cl^{even}(R^3) \oplus CL^{odd}(R^3) & \cong & H \oplus HO \\ \wr\wr & & \wr\wr & & \wr\wr \\ R^8 & \cong & R^4 \oplus R^4 & \cong & R^4 \oplus R^4 \end{array}$$

Next we shall investigate in turn each of the two components in this direct sum.

15.1 ODD ORDER: MASS-POINTS

The ambient geometric space underlying contemporary computer graphics is the 4-dimensional vector space of mass-points [Goldman, 2002]: three of the dimensions are spatial; the fourth dimension is due to the mass (see Section 3.1). Let MP denote the 4-dimensional space of mass-points for points in R^3. We are now going to represent this space of mass-points MP as a 4-dimensional subspace of the 8-dimensional Clifford algebra $Cl(R^3)$ associated with R^3. Since the fourth dimension is mass-like rather than spatial, we should expect that our representation for the fourth mass-like dimension would be qualitatively somewhat different from our representation for the three spatial dimensions.

In the Clifford algebra associated to R^3, every vector v can be expressed uniquely in terms of our fixed orthonormal basis:

$$v = v_1e_1 + v_2e_2 + v_3e_3.$$

But in the standard geometric interpretation of the Clifford algebra for R^3, there is no way to represent points, let alone mass-points. Therefore we shall now adopt a nonstandard interpretation.

We need to represent one more dimension, the dimension corresponding to mass. For this purpose, we will adopt the pseudoscalars. We shall represent the mass m by scalar multiples of the pseudoscalar

$$O = -e_1 e_2 e_3,$$

and we shall identify the pseudoscalar O with the point (or more precisely with the vector in four dimensions representing the point) at the origin of a 3-dimensional coordinate system. Note that with this interpretation O represents a point, not a vector in three dimensions, since the origin of the coordinate system is not the same as the zero vector (see Figure 3.3).

Classically the pseudoscalar $e_1 e_2 e_3$ is used to represent an element of unit volume. Here we are going to use $-e_1 e_2 e_3$ to represent a point. Letting the pseudoscalar $-e_1 e_2 e_3$ represent a point is essentially equivalent to invoking a vector space isomorphism

$$T : Cl(R^3)^{odd} \to MP,$$

where

$$T(e_1) = e_1,\ T(e_2) = e_2,\ T(e_3) = e_3,\ T(-e_1 e_2 e_3) = O.$$

The minus sign is inserted in $-e_1 e_2 e_3$ to make the signs turn out right when we multiply the pseudoscalar O by other elements of the Clifford algebra. For example, by Equations (14.14) and (14.15):

$$O(u \wedge v) = u \times v = (u \wedge v)\, O$$

$$O(u \times v) = u \wedge v = (u \times v)\, O.$$

We shall often abuse notation and use the symbol O to represent both the pseudoscalar $-e_1 e_2 e_3$ in the Clifford algebra $Cl(R^3)$ and the origin the coordinate system in MP. The correct interpretation will be clear from the context.

Now every mass-point p can be represented uniquely in this Clifford algebra as the sum of a vector v and the fixed point O times a mass m—that is,

$$p = m\, O + v.$$

(see again Figure 3.3). This formula means that p has mass m and is located at the point $p / m = O + v / m$. As usual, we shall write

$$m\, O + v \equiv O + v / m$$

to indicate that the mass-point $m\, O + v$ is located at the point $O + v / m$.

To summarize: we shall use the odd dimensional elements of the Clifford algebra to represent mass-points. Elements of dimension one are vectors; elements of dimension three have mass. The sum of an element of dimension 1 and an element of dimension 3 is a mass-point.

An algebra is only an algebra. The formal rules of the Clifford algebra are fixed. But the geometric interpretation we assign to elements of this algebra is completely up to us, constrained only by consistency and applicability.

15.2 EVEN ORDER: QUATERNIONS

We have already observed in Section 14.5 that the even dimensional elements of the Clifford algebra for R^3 are isomorphic to the quaternion algebra H—that is,

$$Cl(R^3)^{even} \cong H.$$

Let π represent multiplication in the Clifford algebra $Cl(R^3)$. Then by Equations (14.1) and (14.2)

$$\pi : Cl(R)^{even} \oplus Cl(R^3)^{odd} \to Cl(R^3)^{odd}$$

or equivalently

$$\pi : H \oplus MP \to MP.$$

Thus we can think of the even dimensional elements of the Clifford algebra—the quaternions—as acting by multiplication on the odd dimensional elements of the Clifford algebra—the mass-points. Notice that this device is the same approach that we took with the Clifford algebra for the plane, where the even dimensional elements—the complex numbers—operate on the odd dimensional elements—the vectors in the plane. Thus under Clifford multiplication,

$$H : MP \to MP.$$

From this point of view, the even dimensional elements of the Clifford algebra—the quaternions H—are operators and the odd dimensional elements of the Clifford algebra—the mass-points MP—are operands. Hence just like the Clifford algebra for R^2 and unlike the complex numbers and quaternions we studied in Part I of this text, the Clifford algebra for R^3 separates operators and operands. Notice that H acts on $MP \cong HO$ on both the left and the right, since O commutes with H, but these actions are not identical, since quaternion multiplication is not commutative.

Two of the goals of Part III are:

i. to show how to use these operators and operands to construct some of the classical transformations—rotations, reflections, and perspective projections—of computer graphics on both points and vectors in three dimensions;
ii. to provide simpler, more intuitive proofs of these constructions than are commonly found in the standard literature on Clifford algebras.

We shall now embark upon our pursuit of these two objectives.

CHAPTER 16

Decomposing Mass-Points Into Two Mutually Orthogonal Planes

Every bivector represents a plane. Indeed, by Proposition 14.1, every bivector b can be decomposed into the wedge product of two vectors u, v. The bivector b represents the plane of vectors spanned by any vectors u, v such that $b = u \wedge v$. The vectors u, v are not unique, but the plane determined by the bivector b is unique.

Relative to any fixed bivector b, the 4-dimensional space of mass-points can be decomposed into two mutually orthogonal planes: the plane of vectors determined by b, and the 2-dimensional subspace of the 4-dimensional space of mass-points orthogonal to the plane determined by b. To avoid confusion between the bivector b and the plane determined by the bivector b, we will denote by $b^{\|}$ the plane of vectors determined by the bivector b. Similarly, we will denote by $b^{\perp}$ the mass-points in the plane orthogonal to b. Thus for any bivector b,

$$\begin{array}{ccc} MP & \cong & b^{\|} \oplus b^{\perp} \\ \wr\!\wr & & \wr\!\wr \\ R^4 & \cong & R^2 \oplus R^2 \,. \end{array}$$

Note that here and elsewhere we use the term *plane* to denote any 2-dimensional subspace of the space of mass-points, rather than a physical plane in R^3.

We are now going to study the geometric effect of multiplying an arbitrary mass-point p in MP by a unit quaternion

$$q(b, \theta) = \cos(\theta) + \sin(\theta)b$$

in H, where b is a unit bivector—that is, where b represents a planar segment with unit area. We shall proceed as in Chapter 5 by investigating the geometric effects of this multiplication in the two mutually orthogonal planes $b^{\|}$ and $b^{\perp}$. The parallels between Chapter 5 and Sections 16.1 and 16.2 are completely intentional. The reader who followed Chapter 5 will have no trouble understanding Sections 16.1 and 16.2. The main point here is to understand how the quaternion techniques in Chapter 5 are reinterpreted in the Clifford algebra of R^3.

16.1 ACTION OF $q(b, \theta)$ ON $b^{\perp}$

The plane $b^{\perp}$ in MP is spanned by the pseudoscalar O and the vector bO. Indeed in the space of mass-points, the vector O representing the origin is orthogonal to every vector in three dimensions (see Figure 3.3), and by Corollary 14.2, the vector bO is orthogonal to every vector in the plane determined by the bivector b. Therefore

$$b^{\perp} = span\{O, bO\}.$$

Moreover, the pseudoscalar O and the vector bO actually form an orthonormal basis for $b^{\perp}$, since

i. O is orthogonal to bO (as a vector in the 4-dimensional vector space of mass-points, the origin O is orthogonal to every vector in R^3; see Figure 3.3)
ii. $|bO| = |b| = 1 \quad \{|u \times v| = \text{area}(u, v) = |u \wedge v| \text{ for all vectors } u, v\}$;
iii. $|O| = 1$ (as a vector in the 4-dimensional space of mass-points, $|O| = 1$; see Figure 3.3).

Multiplication by O is a vector space isomorphism from the subspace of the quaternions H spanned by 1, b to the subspace of the mass-points MP spanned by O, bO. Hence to understand how the unit quaternions $q(b,\theta)$ act on $b^{\perp}$, we first need to understand how multiplication works on the subspace of quaternions spanned by 1, b.

Here is the key observation. By Corollary 14.4,

$$|b| = 1 \Rightarrow b^2 = -1.$$

Therefore the quaternion plane spanned by 1, b is isomorphic to the complex plane. Notice that this observation is the exact analogue of the key observation in Chapter 5.

Now we know how multiplication by unit vectors in the complex plane acts on the complex numbers. Multiplication by the complex number

$$e^{i\theta} = \cos(\theta) + \sin(\theta)i$$

rotates vectors in the complex plane counterclockwise through the angle θ. Analogously, multiplication by the unit quaternion

$$q(b,\theta) = \cos(\theta) + \sin(\theta)b$$

rotates mass-points in the plane $b^{\perp}$ counterclockwise through the angle θ. Indeed, notice that in direct analogy with complex multiplication, we have the following results (see also Lemma 5.1).

Lemma 16.1: *Let b be a unit bivector. Then*

i. $q(b,\theta)\, q(b,\phi) = q(b,\theta + \phi)$

ii. $q(b,\phi)\, q(b,\theta) = q(b,\theta+\phi)$
iii. $q(b,\theta)\, q(b,\phi) = q(b,\phi)\, q(b,\theta)$.

Proof: To prove *i*, we simply apply the fact that $b^2 = -1$ and use the trigonometric identities for the sine and the cosine of the sum of two angles:

i.
$$\begin{aligned} q(b,\theta)\, q(b,\phi) &= (\cos(\theta) + \sin(\theta)b)(\cos(\phi) + \sin(\phi)b) \\ &= (\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) + (\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi))b \\ &= \cos(\theta+\phi) + \sin(\theta+\phi)b \\ &= q(b,\theta+\phi). \end{aligned}$$

ii, iii. Follow immediately from *i*. ♦

Corollary 16.2: *The effect of multiplication by $q(b,\theta)$ on the mass-points in the plane $b^{\perp}$ is just counter-clockwise rotation through the angle θ in the plane $b^{\perp}$. Moreover, since*

$$Ob = bO,$$

multiplication by $q(b,\theta)$ on the left and the right has the same effect on the elements in $b^{\perp}$.
Proof: It is enough to consider mass-points p represented by 4-dimensional vectors of unit length in the plane $b^{\perp} = span\{O, bO\}$. Since O, bO is an orthonormal basis for $b^{\perp}$, there is an angle ϕ such that

$$p = \cos(\phi)O + \sin(\phi)(bO) = q(b,\phi)O.$$

(see Figure 16.1). Therefore by Lemma 16.1

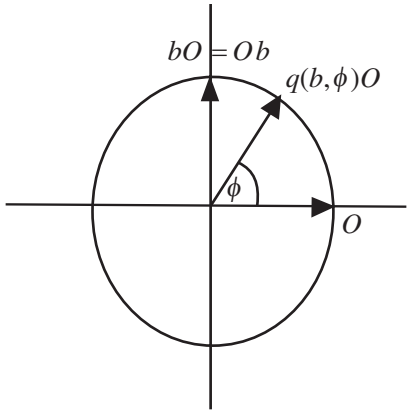
$$q(b,\theta)p = q(b,\theta)q(b,\phi)O = q(b,\theta+\phi)O = \cos(\theta+\phi)O + \sin(\theta+\phi)(bO).$$

Thus the effect of multiplication by $q(b,\theta)$ on the mass-points in the plane $b^{\perp} = span\{O, bO\}$ is just counterclockwise rotation through the angle θ. ♦

16.2 ACTION OF $q(b,\theta)$ ON $b^{\parallel}$

Let v be a unit vector in the plane represented by the bivector b. Then just as O, bO are an orthonormal basis for $b^{\perp}$, the vectors v, bv are an orthonormal basis for $b^{\parallel}$. This observation can be proved in the following fashion.
Lemma 16.3: *Let b be a bivector*, and *let v be a vector in the plane $b^{\parallel}$. Then*

i. $bv = bO \times v$
ii. $vb = v \times bO$



$$b^{\perp} = span\{O, bO\}$$

FIGURE 16.1: The plane $b^{\perp} = span\{O, bO\}$ in the space of mass-points MP. The vector in four dimensions representing the mass-point $q(b, \phi)O$ makes an angle ϕ with the O-axis. Therefore the action of $q(b, \theta)$ on the elements of $b^{\perp}$ is simply counterclockwise rotation by the angle θ. Moreover, since $Ob = bO$, the action of $q(b, \theta)$ on the left and the right is the same.

Proof: By Equation (14.8)

$$bv = (bv)O^2 = ((bO)v)O = (-(bO \cdot v) + (bO \wedge v))O.$$

But

$bO \cdot v = 0$ because by Corollary 14.2 bO is orthogonal to the plane $b^{\|}$

$(bO \wedge v)O = bO \times v$ by Equation (14.14).

Therefore,

$$bv = bO \times v.$$

Similarly,

$$vb = v \times bO. \qquad \blacklozenge$$

Corollary 16.4: *Let b be a bivector*, and *let v be a vector in the plane $b^{\|}$. Then*

i. *bv is a vector in the plane $b^{\|}$*
ii $bv \perp v$
iii. $|bv| = |b|\,|v|$
iv. $vb = -bv$

Thus if $|b| = |v| = 1$, *then* v, bv *is an orthonormal basis for* $b^{\|}$

Proof: These results follow immediately from Lemma 16.3. ♦

Let v be a vector in $b^{\|}$. Using the orthonormal basis v, bv for $b^{\|}$, we can now investigate the effect of multiplication by $q(b, \theta)$ on vectors v in the plane $b^{\|}$.

Corollary 16.5: *The effect of multiplication by* $q(b, \theta)$ *on the left on vectors* v *in the plane* $b^{\|}$ *is just counterclockwise rotation through the angle* θ *in the plane* $b^{\|}$. *Moreover, since*

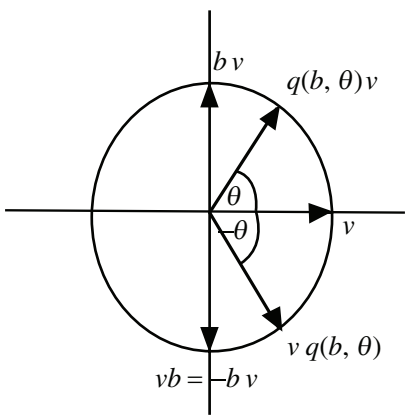
$$vb = -bv,$$

multiplication by $q(b, \theta)$ *on the right on vectors in* $b^{\|}$ *rotates these vectors clockwise through the angle* θ *in the plane* $b^{\|}$.

Proof: It is enough to prove this result for unit vectors v in the plane $b^{\|}$. Now

$$q(b, \theta)v = (\cos(\theta) + \sin(\theta)b)v = \cos(\theta)v + \sin(\theta)(bv).$$

Therefore since by Corollary 16.4 v, bv is an orthonormal basis for $b^{\|}$, multiplication by $q(b, \theta)$ on the left rotates the vector v counterclockwise by the angle θ in the plane $b^{\|}$ (see Figure 16.2). Similarly, since $vb = -bv$, multiplication by $q(b, \theta)$ on the right rotates the vector v clockwise by the angle θ in the plane $b^{\|}$. ♦



$$b^{\|} = span\{v, b\,v\}$$

FIGURE 16.2: The plane $b^{\|} = span\{v, bv\}$ in the space of mass-points MP. The vector $q(b, \theta)v$ makes an angle θ with the v-axis. Therefore the action represented by multiplying vectors v in $b^{\|}$ by $q(b, \theta)$ on the left is simply counterclockwise rotation by the angle θ. Moreover, since $vb = -bv$ the action of $q(b, \theta)$ on the left and the action of $q(b, \theta)$ on the right rotate v by the same amount in opposite directions. Compare with Figure 5.2.

16.3 SANDWICHING

To facilitate our future discussions, we introduce notation analogous to the notation in Chapter 5. Let q be a quaternion and let p be a mass-point. Then

$L_q(p) = qp$	left multiplication by q
$R_q(p) = pq$	right multiplication by q
$T_q(p) = qpq = L_q(R_q(p))$	sandwiching p between two copies of q
$S_q(p) = qpq^* = L_q(R_{q^*}(p))$	sandwiching p between q and q^*.

We now summarize the geometric effects of multiplication by the unit quaternions

$$q(b, \theta) = \cos(\theta) + \sin(\theta)b$$

on vectors in the plane $b^{\parallel}$ and on mass-points in the complementary plane $b^{\perp}$.

Proposition 16.6: Rotation in $b^{\perp}$

Let

- *b = a unit bivector*
- *p = a mass-point in $b^{\perp}$*

Then

i. $L_{q(b,\,\theta)}(p) = q(b, \theta)\, p$ *rotates p by the angle θ in the plane $b^{\perp}$*

ii. $R_{q(b,\,\theta)}(p) = p\, q(b, \theta)$ *rotates p by the angle θ in the plane $b^{\perp}$*

Proof: These results follow immediately from Corollary 16.2. ♦

Proposition 16.7: Rotation in $b^{\parallel}$

Let

- *b = a unit bivector*
- *v = a vector in $b^{\parallel}$*

Then

i. $L_{q(b,\,\theta)}(v) = q(b, \theta)\, v$ *rotates v by the angle θ in the plane $b^{\parallel}$*

ii. $R_{q(b,\,\theta)}(v) = vq(b, \theta)$ *rotates v by the angle $-\theta$ in the plane $b^{\parallel}$*

Proof: These results follow immediately from Corollary 16.5. ♦

It follows by Propositions 16.6 and 16.7 that multiplication by a unit quaternion $q(b, \theta)$ represents a double isoclinic rotation on the space of mass-points (see Figure 5.3), since mass-points in the two mutually orthogonal planes $b^{\parallel}$ and $b^{\perp}$ are rotated through the same angle θ, although the orientation of the rotation may differ depending on whether the multiplication is on the left or the right. Therefore, just as in Chapter 5, we can now use sandwiching to balance these clockwise and counterclockwise rotations and generate simple rotations in the space of mass-points.

Corollary 16.8: *Sandwiching in* $\boldsymbol{b}^{\perp}$

Let

- b = *a unit bivector*
- p = *a mass-point in* $b^{\perp}$

Then

i. $T_{q(b,\,\theta)}(p) = q(b, \theta)\, p\, q(b, \theta)$ *rotates* p *by the angle* 2θ *in the plane* $b^{\perp}$

ii. $S_{q(b,\,\theta)}(p) = q(b, \theta)\, p\, q^{*}(b, \theta)$ *is the identity on* p.

Proof: These results follow immediately from Proposition 16.6 and Equation (14.24). ■

Corollary 16.9: *Sandwiching in* $\boldsymbol{b}^{\parallel}$

Let

- b = *a unit bivector*
- v = *a vector in* $b^{\parallel}$

Then

i. $T_{q(b,\,\theta)}(v) = q(b, \theta)\, v\, q(b, \theta)$ *is the identity on* v

ii. $S_{q(b,\,\theta)}(v) = q(b, \theta)\, v\, q^{*}(b, \theta)$ *rotates* v *by the angle* 2θ *in the plane* $b^{\parallel}$.

Proof: These results follow immediately from Proposition 16.7 and Equation (14.24). ♦

By Corollaries 16.8 and 16.9, the sandwiching operators $S_{q(b,\,\theta)}(v)$ and $T_{q(b,\,\theta)}(p)$ represent simple rotations—that is, rotations in a single plane—in the 4-dimensional space of mass-points $MP = b^{\parallel} \oplus b^{\perp}$. In the next chapter we shall use these two sandwiching operators to study rotations, reflections, and perspective projections in R^3.

Exercises:

16.1. Let b be a bivector and let p be a mass-point. Show that

$$T_{-b}(p) = T_b(p).$$

16.2. Let u be a unit vector, and define

$$q(u, \theta) = \cos(\theta)O + \sin(\theta)u$$

$$q^{*}(u, \theta) = \cos(\theta)O - \sin(\theta)u$$

a. Show that

i. $T_{q(b,\,\theta)}(v) = q(bO, \theta)v\, q(bO, \theta) = T_{q(b,\,\theta)O}(v)$

ii. $S_{q(b,\,\theta)}(v) = q(bO, \theta)v\, q^{*}(bO, \theta) = S_{q(b,\,\theta)O}(v)$.

b. Conclude that Corollaries 16.8 and 16.9 remain valid when we replace the bivector b by the normal vector bO.

• • • •

CHAPTER 17

Rotation, Reflection, and Perspective Projection

Let us pause here for a moment to compare our sandwiching results in Section 16.3 with our sandwiching results in Chapter 5. Let b be the bivector representing a plane $b^{\|}$ in R^3, and let $N = b\,O$ be the unit vector in R^3 perpendicular to the plane $b^{\|}$. Comparing the results in Section 16.3 to the results in Chapter 5, we see that the plane $\boldsymbol{b}^{\perp}$ plays the same role as the plane O, N and the plane $\boldsymbol{b}^{\|}$ plays the same role as the plane perpendicular to O, N. In particular, sandwiching mass-points in the planes $\boldsymbol{b}^{\perp}$ and $\boldsymbol{b}^{\|}$ with the unit quaternions

$$q(b,\theta) = \cos(\theta) + \sin(\theta)\, b$$

has the same effect as sandwiching the mass-points in the plane of O, N and the plane perpendicular to O, N with the unit quaternions

$$q(N,\theta) = \cos(\theta)\, O + \sin(\theta) N.$$

Thus our proofs of the sandwiching formulas for rotation, reflection, and perspective projection in the quaternion model presented in Chapter 6 can be transposed almost line for line to formulas for rotation, reflection, and perspective projection in the Clifford algebra for R^3.

Applying sandwiching to compute rotation and mirror image on vectors in R^3 is a well-known technique in Clifford algebra [Dorst *et al.*, 2007]. Since our model of Clifford algebra includes points as well as vectors, here we shall simply extend these standard results on rotation and reflection from vectors to points. Our proofs, however, are simpler than the standard proofs, since we shall take advantage of what we already know about the simple effects of the sandwiching maps $T_{q(b,\,\theta)}$ and $S_{q(b,\,\theta)}$ on the planes $b^{\|}$ and $b^{\perp}$. Using techniques from Clifford algebra to compute perspective projection seems to be new. We are able to employ sandwiching to perform perspective projection in R^3 only because we have adopted a rather unconventional interpretation of the pseudoscalars, an interpretation that allows us to incorporates mass-points into the Clifford algebra of R^3.

17.1 ROTATION

Rotations in three dimensions are typically specified by an axis of rotation and an angle of rotation. But in three dimensions planes are dual to vectors, so instead of specifying an axis of rotation, we can specify a plane of rotation, a plane perpendicular to the axis of rotation. Rotation occurs mostly in this rotation plane, since a vector is rotated around an axis of rotation by rotating its orthogonal projection in the plane of rotation. In Clifford algebra, planes are represented by bivectors. Therefore here we shall specify a rotation by a bivector b and an angle θ. The plane of rotation is $b^{||}$, and the axis of rotation is the vector bO, which by Corollary 14.2 is orthogonal to the rotation plane $b^{||}$.

Theorem 17.1: *Sandwiching with conjugate quaternions rotates vectors in 3-dimensions*

Let

- b = *a unit bivector*
- v = *a vector in 3-dimensions*

Then

- $S_{q(b,\theta/2)}(v) = q(b,\theta/2)\, v\, q^*(b,\theta/2)$ *rotates v by the angle θ around the axis bO.*

Proof: Let $v_{||}$ be the component of v in $b^{||}$, and let $v_{\perp}$ be the component of v perpendicular to $b^{||}$. Then

- $S_{q(b,\theta/2)}(v_{\perp}) = q(b,\theta/2)\, v_{\perp}\, q^*(b,\theta/2) = v_{\perp}$ (Corollary 16.8)
- $S_{q(b\,\theta/2)}(v_{||}) = q(b,\theta/2)\, v_{||}\, q^*(b,\theta/2) = \cos(\theta)\, v_{||} + \sin(\theta)\, b\, v_{||}$
  rotates $v_{||}$ by the angle θ in the plane $b^{||}$. (Corollary 16.9)

Therefore since $S_{q(b,\,\theta/2)}$ is a linear transformation and $v = v_{\perp} + v_{||}$,

$$S_{q(b,\,\theta/2)}(v) = S_{q(b,\,\theta/2)}(v_{\perp}) + S_{q(b,\,\theta/2)}(v_{||}) = v_{\perp} + \cos(\theta)v_{||} + \sin(\theta)bv_{||}.$$

Hence by Equation (6.1) (where we have interchanged the roles of $v_{\perp}$ and $v_{||}$ because in Equation (6.1) we are considering the parallel and perpendicular components of v with respect to the axis of rotation rather than the components of v relative to the plane of rotation) sandwiching has the same effect on v as rotating v in 3-dimensions by the angle θ around the axis vector bO. ♦

Theorem 17.2: *Sandwiching with conjugate quaternions also rotates points in 3-dimensions*

Let

- b = *a unit bivector*
- P = *a point in 3-dimensions*

Then

- $S_{q(b,\,\theta/2)}(P) = q(b,\,\theta/2)\, P\, q^*(b,\,\theta/2)$ *rotates P by the angle θ around the line passing through the point O parallel to the vector bO.*

Proof: Since P be a point in affine space,

$$P = O + (P - O).$$

But

- $S_{q(b,\,\theta/2)}(O) = O$ (Corollary 16.8)
- $S_{q(b\,\theta/2)}(P - O)$ *rotates the vector* $P - O$ *by the angle* θ *around the axis* bO (Theorem 17.1).

Therefore since $S_{q(b,\,\theta/2)}$ is a linear transformation and $P = O + (P - O)$,

$$S_{q(b,\,\theta/2)}(P) = O + S_{q(b,\,\theta/2)}(P - O).$$

Hence by Theorem 17.1 and Equation (6.2), sandwiching has the same effect on P as rotating P in 3-dimensions by the angle θ around the line passing through the point O parallel to the vector bO. ♦

Exercises:

17.1. Let $q_1 = q(b_1, \theta_1 / 2)$, $q_2 = q(b_2, \theta_2 / 2)$, and let p be a mass-point.

a. Show that

$$(S_{q_1} \circ S_{q_2})(p) = S_{q_1 q_2}(p).$$

b. Conclude from part a that composing rotations is equivalent to multiplying quaternions.

17.2. Using Exercise 16.2, show that Theorems 17.1 and 17.2 remain valid when we replace the bivector b by the normal vector bO.

17.2 MIRROR IMAGE

Mirror images in 3-dimensions are typically specified by a vector normal to the mirror plane. But in 3-dimensions planes are dual to vectors, so instead of specifying a normal vector, we can specify the mirror plane. In Clifford algebra, planes are represented by bivectors. Therefore just as with rotations, here we shall specify reflections by bivectors b. The mirror plane is $b^{\|}$ and the normal vector is bO, which is orthogonal to the mirror plane $b^{\|}$.

Theorem 17.3: *Sandwiching with unit bivectors reflects vectors in 3-dimensions*

Let

- b = *a unit bivector*
- v = *a vector in 3-dimensions*

Then

- $T_b(v) = b\, v\, b = -S_b(v)$ *is the mirror image of* v *in the plane* $b^{\|}$.

Proof: Let $v_{\|}$ be the component of v in $b^{\|}$, and let $v_{\perp}$ be the component of v perpendicular to $b^{\|}$. Since $b = \cos(\pi/2) + \sin(\pi/2)\, b = q(b,\pi/2)$:

- $T_b(v_{\perp}) = b\, v_{\perp}\, b = q(b,\pi/2)\, v_{\perp}\, q(b,\pi/2) = T_{q(b,\pi/2)}(v_{\perp}) = -v_{\perp}$ (Corollary 16.8)
- $T_b(v_{\|}) = b\, v_{\|}\, b = q(b,\pi/2)\, v_{\|}\, q(b,\pi/2) = T_{q(b,\pi/2)}(v_{\|}) = v_{\|}$ (Corollary 16.9)

Therefore since T_b is a linear transformation and $v = v_{\|} + v_{\perp}$,

$$T_b(v) = T_b(v_{\|}) + T_b(v_{\perp}) = v_{\|} - v_{\perp}.$$

Hence by Equation (6.3) (where again we have interchanged the roles of $v_{\perp}$ and $v_{\|}$ because in Equation (6.3) we are considering the parallel and perpendicular components of v with respect to the normal to the mirror plane rather than the components of v relative to the mirror plane) sandwiching with b has the same effect on v as reflecting v in 3-dimensions in the plane $b^{\|}$. ♦

Theorem 17.4: *Sandwiching* $\boldsymbol{P-2O}$ *with the bivector* b *reflects points* P *in the plane* $\boldsymbol{b}^{\|}$ *passing through the point* O

Let

- b = *a unit bivector*
- P = *a point in 3-dimensions*

Then

- $T_b\,(P-2O) = b\,(P-2O)b$ *is the mirror image of the point* P *in the plane* $b^{\|}$ *passing through the point* O.

Proof: Clearly,

$$(P-2O) = (P-O) - O.$$

But

- $T_b\,(-O) = -\,b\,O\,b = -b^2 O = O$
- $T_b\,(P-O) = b\,(P-O)b$

Therefore since T_b is a linear transformation and $P-2O = (P-O) - O$,

$$T_b\,(P-2O) = O + b\,(P-O)b,$$

which by Theorem 17.3 and Equation (6.4) is the mirror image of the point P in the plane $\boldsymbol{b}^{\|}$ passing through the point O. ♦

Exercises:

17.3. Using Exercise 16.2, show that Theorems 17.3 and 17.4 remain valid when we replace the bivector b by the normal vector bO.

17.4. Let b_1, b_2 be bivectors.

a. Show that
   i. $b_1 b_2 = (b_2 b_1)^*$
   ii. $(T_{b_2} \circ T_{b_1})(v) = S_{b_2 b_1}(v)$.

b. Conclude from part a that the composite of two reflections is equivalent to one rotation.

17.5. Let b be a unit bivector, and let u, v be two unit vectors in the plane $b^{\|}$ chosen so that the angle between u and v is $\theta/2$.

a. Show that
   i. $(-vO)(uO) = (v \cdot u) - v \wedge u = \cos(\theta/2) + \sin(\theta/2)\, b = q(b, \theta/2)$.
   ii. $(uO)(-vO) = (v \cdot u) - u \wedge v = \cos(\theta/2) - \sin(\theta/2)\, b = q^*(b, \theta/2)$.

b. Conclude from part a that
   $S_{q(b,\, \theta/2)}(w) = (T_{-vO} \circ T_{uO})(w) = (\mathrm{T}_{vO} \circ T_{uO})(w)$.

c. Conclude from part b that every rotation is the composite of two reflections.

17.3 PERSPECTIVE PROJECTION

A perspective plane in 3-dimensions is typically specified by a point on the plane and a vector normal to the plane. But once again in Clifford algebra planes are dual to vectors, so instead of specifying a normal vector, we can specify a perspective plane using a bivector. Therefore here we shall specify a perspective plane by a bivector b. The perspective plane is $b^{\|}$ and the normal vector is bO, which is orthogonal to the perspective plane $b^{\|}$.

Theorem 17.5: *Sandwiching vectors to the eye between two copies of a unit quaternion gives perspective*

Suppose that $0 < \theta < \pi$ *and let*

- $b =$ *a unit bivector*
- $E = O + (\cot(\theta) - \csc(\theta))bO =$ *eye point*
- $P =$ *a point in three dimensions*

Then

- $T_{q(b,-\theta/2)}(P - E) = q(b,-\theta/2)(P - E)\, q(b,-\theta/2)$ *is a mass-point, where*
  —*the point is located at the perspective projection of the point* P *from the eye point* E *onto the plane* $b^{\|}$ *passing through the point* $O + \cot(\theta)(bO) \equiv T_{q(b,\, -\theta/2)}(bO)$.

—*the mass is equal to* $d\sin(\theta)$, *where* d *is the distance of the point* P *from the plane* $b^{\parallel}$ *passing through the eye point* E.

Proof: Let $P - E = d(bO) + v$, where d is a scalar and $v \perp bO$ is a vector in the plane $b^{\parallel}$ (see Figure 17.1). Then since by Corollary 16.8 the map $T_{q(b,-\theta/2)}$ rotates mass-points in the plane $b^{\perp} = span\{O, bO\}$ by the angle $-\theta$:

- $T_{q(b,-\theta/2)}(d(bO)) = d\cos(\pi/2 - \theta)O + d\sin(\pi/2 - \theta)(bO) = d\sin(\theta)O + d\cos(\theta)(bO)$

Moreover by Corollary 16.9, the map $T_{q(b,-\theta/2)}$ is the identity on vectors in the plane $b^{\parallel}$, so

- $T_{q(b,-\theta/2)}(v) = v.$

Therefore by linearity

- $$T_{q(b,-\theta/2)}(P-E) = T_{q(b,-\theta/2)}(d(bO)+v) = d\sin(\theta)O + d\cos(\theta)(bO) + v$$
$$\equiv O + \cot(\theta)(bO) + \csc(\theta)\frac{v}{d}.$$

Since by construction

- $O + \cot(\theta)(bO) = E + \csc(\theta)(bO),$

it follows by similar triangles (see Figure 17.1) that the point corresponding to the mass-point $T_{q(b,-\theta/2)}(P-E)$ is the perspective projection of the point P from the eye point E onto the plane $b^{\parallel}$ passing through the point $O + \cot(\theta)bO$. Moreover, the mass is $d\,sin(\theta)$, where d is the distance of the point P from the plane $b^{\parallel}$ passing through the eye point E. ♦

As in Section 6.3, we shall call the position of the eye point E and the plane of projection $b^{\parallel}$ in Theorem 17.5 the canonical position for the angle θ and the bivector b, and we shall write

$E(b, \theta) = O + (\cot(\theta) - \csc(\theta))bO$
$b^{\parallel}(\theta) =$ *plane of projection parallel to* $b^{\parallel}$ *at a distance* $d = csc(\theta)$ *from the eye point* $E(b,\theta)$ *along the normal vector* bO.

Now, as in Section 6.3, Theorem 17.5 remains valid even if the eye point E and the plane of projection $b^{\parallel}$ are not in canonical position.

Theorem 17.6: *Sandwiching vectors to the eye with* $q(N, -\theta/2)$ *gives perspective*
Suppose that $0 < \theta < \pi$, and *let*

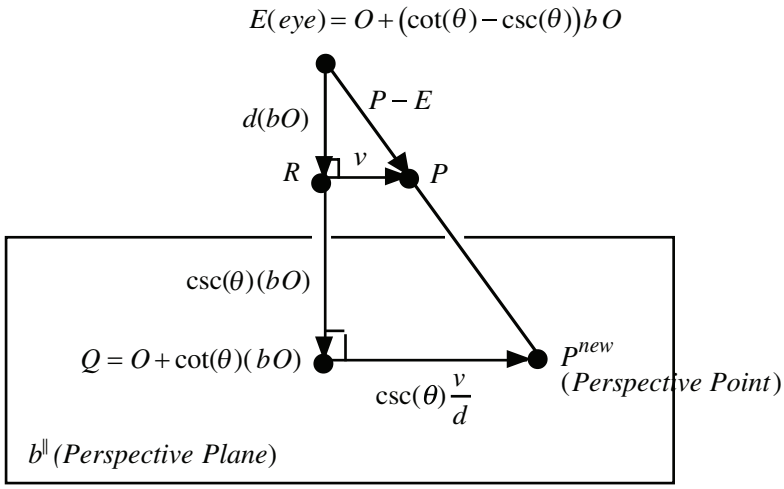


FIGURE 17.1: By similar triangles, the point

$$P^{new} = O + \cot(\theta)(bO) + \csc(\theta)\frac{v}{d} = E + \csc(\theta)(bO) + \csc(\theta)\frac{v}{d}$$

is the perspective projection of the point P from the eye point E onto the plane $b^{\parallel}$ passing through the point $Q = O + \cot(\theta)(bO)$.

- *E = eye point*
- *b = a unit bivector*
- *$b^{\parallel}$ = plane of projection parallel to $b^{\parallel}$ at a distance $d = \csc(\theta)$ from the eye point E along the normal vector bO*
- *P = a point in 3-dimensions*

Then

- *$T_{q(b,-\theta/2)}(P - E) = q(b, -\theta/2)(P - E)\, q(b, -\theta/2)$ is a mass-point, where:*

  —*the point is located at the perspective projection of the point P from the eye point E onto the plane $b^{\parallel}$ at a distance $d = \csc(\theta)$ from the eye point E along the normal vector bO, translated by the vector $E(b, \theta) - E$ to the canonical plane $b^{\parallel}(\theta)$*

  —*the mass is equal to $d\sin(\theta)$, where d is the distance of the point P from the plane $b^{\parallel}$ passing through the eye point E.*

Proof: This result follows immediately from Theorem 17.5 and the fact that perspective projection commutes with translation. ♦

Exercise:

17.6. Using Exercise 16.2, show that Theorems 17.5 and 17.6 remain valid when we replace the bivector b by the normal vector bO—that is when we replace $T_{q(b,\,-\theta/2)}(P-E)$ by $T_{q(bO,\,-\theta/2)}(P-E)$.

CHAPTER 18

Summary

In the standard model of Clifford algebra for 3-dimensions the basic operations are reflections (versors) [Dorst *et al.*, 2007]: every rotation (rotor) in 3-dimensions is the product of two reflections in 3-dimensions. In our homogeneous model of Clifford algebra, rotations in 3-dimensions still factor into pairs of reflections in 3-dimensions (see Exercise 17.5). Nevertheless, in our homogeneous model, the basic operations are not reflections in 3-dimensions, but rather simple rotations in 4-dimensions.

For any bivector b, rotations in 4-dimensions that leave the plane $b^{\perp}$ fixed correspond to rotations in 3-dimensions in the plane $b^{\parallel}$; rotations in 4-dimensions that leave the plane $b^{\parallel}$ fixed correspond to either reflections or perspective projections in 3-dimensions. Thus in our homogeneous model of Clifford algebra rather than to think of reflections as the basic operations, it is more natural to think of reflections in 3-dimensions as special rotations in 4-dimensions. The main strength of our technique is that this approach allows us to model perspective projections in 3-dimensions as orthogonal transformations in 4-dimensions.

The main weakness of our method is that our approach does not model translations. In the conformal model of Clifford algebra [Dorst *et al.*, 2007], every translation is the product of two reflections in parallel planes. But in our model there is no way to represent translation. Indeed our model is not translation invariant; the pseudoscalar $O = -e_1e_2e_3$ is rotation invariant (see Exercise 14.3), but O is not translation invariant. In effect, we have traded the capacity to perform translations for the ability to perform perspective projections.

Below we summarize our principal insights along with our main results. Many of these ideas have actually been around for a long time; our main contribution is rethinking these ideas and reassembling them in new and useful ways. The reader should compare these insights and results for Clifford algebra to the parallel insights and results for quaternions outlined in Chapter 7.

1. *The pseudoscalars can be exploited to represent mass.*
   a. In the standard geometric interpretation of the Clifford algebra for R^3, the pseudoscalar $e_1e_2e_3$ represents a unit volume. Here we have adopted an alternative interpretation, exploiting the pseudoscalars to represent mass instead of volume.

2. *The Clifford algebra for R^3 decomposes into two disjoint 4-dimensional vector spaces: the space of mass-points and the space of quaternions.*
   a. We have the following vector space isomorphisms:

$$\begin{array}{ccccc} Cl(R^3) & \cong & Cl^{even}(R^3) \oplus CL^{odd}(R^3) & \cong & H \oplus MP \\ \cong & & \cong & & \cong \\ R^8 & \cong & R^4 \oplus R^4 & \cong & H \oplus HO \end{array}$$

3. *For any unit bivector b, the planar subset $C_b = span\{1,b\}$ of the quaternions H is isomorphic to the complex plane C.*
   a. $b^2 = -1 \Rightarrow C_b \cong C$

4. *For any bivector b, we can decompose the 4-dimensional space of mass-points into two mutually orthogonal planes: the vectors in $b^{\|}$ and the mass-points in $b^{\perp}$.*
   a. In particular, we have

$$\begin{array}{ccc} MP & \cong & b^{\|} \oplus b^{\perp} \\ \cong & & \cong \\ R^4 & \cong & R^2 \oplus R^2 \end{array}$$

5. *The plane $b^{\|}$ has special properties.*
   a. $b^{\|}$ represents a plane of vectors in R^3, the vectors in the plane of the bivector b.
   b. $b^{\|} = C_b v$ for any vector v in $b^{\|}$.
      i. Left multiplication by $q(b,\theta)$ represents counterclockwise rotation by θ in $b^{\|}$.
      ii. Right multiplication by $q(b,\theta)$ represents clockwise rotation by θ in $b^{\|}$.
         (a) $vb = -bv \Rightarrow vq(b, \theta) = q^*(b, \theta)v = q(b,-\theta)v$.

6. *The plane $b^{\perp}$ has special properties.*
   a. $b^{\perp}$ represents a line of points in R^3, the line through the point O in the direction of the vector bO.
   b. $b^{\perp} = C_b O$
      i. Left multiplication by $q(b, \theta)$ represents counterclockwise rotation by θ in $b^{\perp}$.
      ii. Right multiplication by $q(b, \theta)$ represents counterclockwise rotation by θ in $b^{\perp}$.
         (a) $Ob = bO \Rightarrow Oq(b, \theta) = q(b,\theta)O$.

7. *Sandwiching is the fundamental operation of quaternions on mass-points.*
   a. Sandwiching with unit quaternions represents simple rotations on the 4-dimensional space of mass-points, unlike left and right multiplication by quaternions, which represent double isoclinic rotations on the 4-dimensional space of mass-points.

b. $S_{q(b,\,\theta)}$ is the identity on $b^{\perp}$ and rotates vectors in $b^{\parallel}$ by the angle 2θ.

   i. $S_{q(b,\,\theta)}$ is the identity on the line through the point O in the direction of the vector bO.

   ii. $S_{q(b,\,\theta)}$ rotates vectors by the angle 2θ around the axis bO.

c. $T_{q(b,\,\theta)}$ is the identity on $b^{\parallel}$ and rotates mass-points in $b^{\perp}$ by the angle 2θ.

   i. $T_{q(b,\,\theta)}$ is the identity on the plane of vectors determined by the bivector b.

   ii. $T_b = T_{q(\mathrm{b},\,\pi/2)}$ reflects vectors in the plane $b^{\parallel}$.

   iii. $T_{q(b,\,\theta)}$ represents perspective projection into a plane parallel to $b^{\parallel}$.

• • • •

CHAPTER 19

Some Simple Alternative Homogeneous Models for Computer Graphics

The homogeneous model presented here for computer graphics uses the full 8-dimensional Clifford algebra for R^3: the even dimensional elements, the quaternions, are the operators, and the odd dimensional elements, which we interpret as mass-points, are the operands. There are, however, other perhaps even simpler alternatives to this basic model.

One natural alternative is to employ the Clifford algebra for R^4. There are several potential advantages to this homogeneous model. First, we do not need to adopt the pseudoscalars to represent mass; rather since there are now four canonical basis vectors e_1, e_2, e_3, e_4, we can simply use scalar multiples of e_4 to represent mass. This approach is indeed standard practice in computer graphics.

Second, instead of four independent bivectors, we now have six independent bivectors: e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4. The bivectors b spanned by e_1e_2, e_2e_3, e_3e_1 can be factored as in Proposition 14.1 into the wedge product of two vectors u, v in R^3. Thus just as in the Clifford algebra for R^3, the maps $S_{q(b,\theta)}$ represent simple rotations in the plane $b^{\|}$ in R^3, since it is straightforward to check that these maps are the identity on the plane in R^4 spanned by $u \times v$ and e_4 (see Exercise 19.1). Furthermore, if $b = e_4v$ for any unit vector v in R^3, then the maps $S_{q(b,\theta)}$ represent simple rotations in the plane $b^{\|}$ in R^4 determined by e_4 and v, since once again it is easy to verify that these maps are the identity on the plane of vectors in R^3 perpendicular to v. Thus if $b = e_4v$, then the map $S_{q(b,\theta)}$ represents perspective projection into a plane perpendicular to v (see Exercise 19.2). Notice that we no longer need the sandwiching maps $T_{q(b,\theta)}$ to represent perspective projections; all the simple rotations we need are now given by the sandwiching maps $S_{q(b,\theta)}$ for different choices of the bivector b.

The main drawback to using the Clifford algebra for R^4 as our homogeneous model is that we must now work in a 16-dimensional rather than an 8-dimensional vector space. So what we could gain in conceptual simplicity, we might lose in computational efficiency. Nevertheless, this richer, higher dimensional model with six independent bivectors may yet allow us to represent additional linear, affine, or projective transformations not directly available in the Clifford algebra for

R^3 by using the maps $S_{q(b,\theta)}$ with arbitrary bivectors b. Whether we can actually capture additional useful transformations with this homogeneous model remains a topic for future research.

Instead of going to higher dimensions, we could try to drop down to lower dimensions. We could, for example, use only the odd dimensional elements of the Clifford algebra for R^3 to model the operands, the mass-points, and the sandwiching operators. For any quaternion q, let $\tilde{q}$ denote either q or q^*. Then since $O^2 = 1$ and O commutes with every element in the Clifford algebra,

$$q\,p\,\tilde{q} = (qO)\,p\,(\tilde{q}O).$$

But by duality, if q is a quaternion, then qO, $\tilde{q}O$ are mass-points. Thus instead of sandwiching a mass-point p between two quaternions, we can sandwich p between two mass-points and get the same results (see Exercise 16.2). For example,

$$b\,v\,b = (bO)\,v\,(bO),$$

so to compute the mirror image of a vector v in the plane $b^{\|}$, instead of sandwiching the vector v between two copies of the bivector b, we can sandwich the vector v between two copies of the normal vector bO. In a similar way, all the sandwiching operators for rotation, reflection, and perspective projection can be represented by the mass-points $q(b,\theta)O$ instead of by the quaternions $q(b,\theta)$ (see Exercises 17.2, 17.3, and 17.6). This approach reduces the vector space of operators and operands from 8-dimensions to 4-dimensions, since the operators and operands now reside in the same space. However this model may be a bit awkward to implement, since intermediate products such as $(q(b,\theta)O)p$ are quaternions and so do not reside in the 4-dimensional space of mass-points.

Perhaps a better alternative, and one that has been used many times before (but only for vectors and not for points), is to model both the operators and the operands as quaternions. To do so, we map each mass-point to its dual quaternion, then sandwich the dual using quaternion multiplication, and finally convert the resulting quaternion back to a mass-point by taking its dual. This approach really amounts to just the following identity:

$$q\,p\,\tilde{q} = (q(pO)\,\tilde{q})O.$$

If p is a mass-point and q is a quaternion, then pO is a quaternion and the product $q(pO)\,\tilde{q}$ can be computed using only quaternion multiplication, since each factor in this product is a quaternion. But instead of computing all these duals, we can simply redefine the geometric meaning of the quaternions themselves to represent mass-points. In this interpretation, the scalars are used to represent mass, and the bivectors are interpreted to represent vectors, the vectors dual to the planes originally represented by these bivectors. That is, we invoke the vector space isomorphism

$$T(e_2e_3) = i,\ T(e_3e_1) = j,\ T(e_1e_2) = k,\ T(1) = O,$$

where O is the origin of the coordinate system and i, j, k are unit vectors along the coordinate axes. This interpretation is actually the standard interpretation of quaternions in computer graphics; we use exactly this interpretation in Part I of this monograph.

Remember: *An algebra is only an algebra. The formal rules of the algebra are fixed. But the geometric interpretation we assign to elements of this algebra is completely up to us, constrained only by consistency and applicability.* Nothing but tradition and convention prevents us from interpreting the bivectors as vectors and the scalars as mass. But by invoking this interpretation, we can avoid computing any duals, and all of our computations are confined to the 4-dimensional quaternion algebra. This 4-dimensional homogeneous model for computer graphics may be intellectually less satisfying, but it is certainly computationally more efficient than the 8-dimensional model of operators and operands in the Clifford algebra for R^3. So we have come full circle, back again to the quaternion model we originally presented in Part I before our excursion in Part III into Clifford algebra.

Exercises:

19.1. Consider the Clifford algebra for R^4.

a. Show that if $b = u \wedge v$, where u and v are unit vectors in R^3, then the map $S_{q(b,\theta)}$ is the identity on the plane in R^4 spanned by $u \times v$ and e_4.

b. Using part a, show that the maps $S_{q(b,\theta)}$ represent rotations in the plane $b^{\|}$ in R^3.

19.2. Consider the Clifford algebra for R^4.

a. Show that if $b = e_4v$, where v is a unit vector in R^3, then the map $S_{q(b,\theta)}$ is the identity on the plane of vectors in R^3 perpendicular to v.

b. Using part a, show that the maps $S_{q(b,\theta)}$ represent perspective projections into a plane $b^{\|}$ in R^3.

(Hint: Examine Theorem 17.6 to find the eye point E and the distance from the eye point E to the plane of projection.)

References

Conway, J., and Smith, D. (2003), *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*, A.K. Peters, Natick, MA.

Dorst, L., Fontijne, D., and Mann, S. (2007), *Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry*, Morgan Kaufmann, Amsterdam.

Du Val, P. (1964), *Homographies, Quaternions and Rotations*, Oxford Mathematical Monographs, Clarendon Press, Oxford.

Farouki, R. (2008), *Pythagorean Hodograph Curves: Algebra and Geometry Inseparable*, Springer, Berlin.

Foley, J., van Dam, A., Feiner, S., and Hughes, J. (1990), *Computer Graphic: Principles and Practice, Second Edition*, Addison-Wesley, Reading, MA.

Goldman, R. (2002), On the algebraic and geometric foundations of computer graphics, *Transactions on Graphics*, Vol. 21, pp. 1–35.

Goldman, R. N. (2010), Understanding Quaternions, accepted to appear in *Graphical Models*.

Hamilton, W. R. (1866), *Elements of Quaternions*, Cambridge University Press, Cambridge, UK.

Hanson, A. (2006), *Visualizing Quaternions*, Morgan Kaufmann, New York.

Hurwitz, A. (1898), Uber die Composition der Quadratischen Formen von Beliebig Vielen Variablen, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, Bd. II, pp. 309–316.

Mebius, J. E. (2005), A matrix based proof of the quaternion representation theorem for four-dimensional rotations, http://arXiv:math/0501249v1 [math.GM].

Shoemake, K. (1985), Animating Rotation with Quaternion Curves, *SIGGRAPH '85: Proceedings of the 12th Annual Conference on Computer Graphics and Interactive Techniques*, ACM Press, pp. 245–254. doi:10.1145/325334.325242

Shoemake, K. (1991), Quaternions and 4 × 4 matrices, in *Graphics Gems II*, Edited by J. Arvo, pp. 351–354, Academic Press, New York.

Further Reading

Altmann, S. L. (1986), *Rotations, Quaternions, and Double Groups*, Clarendon Press, Oxford.

Doran, C., and Lasenby, A. (2003), *Geometric Algebra for Physicists*, Cambridge University Press, Cambridge, UK.

Kuipers, J. B. (1998), *Quaternions and Rotation Sequences*, Princeton University Press, Princeton, NJ.

Moore, C. L. (1918), Rotations in hyperspace, *Proceedings of the American Academy of Arts and Sciences*, Vol. 53, pp. 649–694.

Needham, T. (1997), *Visual Complex Analysis*, Oxford University Press, Oxford.

Perwass, C. (2009), *Geometric Algebra With Applications in Engineering*, Springer-Verlag, Berlin.

Author Biography

Ron Goldman is a professor of computer science at Rice University in Houston, Texas. Dr. Goldman received his B.S. in mathematics from the Massachusetts Institute of Technology in 1968 and his M.A. and Ph.D. in mathematics from Johns Hopkins University in 1973. Dr. Goldman's current research interests lie in the mathematical representation, manipulation, and analysis of shape using computers. His work includes research in computer-aided geometric design, solid modeling, computer graphics, polynomials, and splines. He is particularly interested in algorithms for polynomial and piecewise polynomial curves and surfaces, as well as in applications of algebraic and differential geometry to geometric modeling. His most recent focus is on the uses of quaternions and Clifford algebras in computer graphics. Dr. Goldman has published over a hundred articles in journals, books, and conference proceedings on these and related topics. He has also published two books on computer graphics and geometric modeling: *Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling* and *An Integrated Introduction to Computer Graphics and Geometric Modeling.* Dr. Goldman is currently an associate editor of *Computer Aided Geometric Design*. Before returning to academia, Dr. Goldman worked for 10 years in industry solving problems in computer graphics, geometric modeling, and computer-aided design. He served as a mathematician at Manufacturing Data Systems Inc., where he helped to implement one of the first industrial solid modeling systems. Later, he worked as a senior design engineer at Ford Motor Company, enhancing the capabilities of their corporate graphics and computer-aided design software. From Ford, he moved on to Control Data Corporation, where he was a principal consultant for the development group devoted to computer-aided design and manufacture. His responsibilities included database design, algorithms, education, acquisitions, and research. Dr. Goldman left Control Data Corporation in 1987 to become an associate professor of computer science at the University of Waterloo in Ontario, Canada. He joined the faculty at Rice University in Houston, Texas, as a professor of computer science in July 1990.